ON A CLASS OF SEMIGROUP-RINGS

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In this paper the structure and some properties of an arbitrary semigroup-ring $R(s)$ have been investigated where R is a ring with an identity element.

§1. The author (Fotedar 1974) introduced the semigroup-ring $Z(S)$ where, $S = \{b^m a^n \mid m, n \in N$ (the set of all non-negative integers)$\}$ is a semigroup under the multiplication rule,

$$b^m a^n . b^{m'} a^{n'} = \begin{cases} b^m a^{n-m'+n'}, \text{ if } n \geqslant m', \\ b^{m'-n+m} a^{n'}, \text{ if } n \leqslant m', \end{cases}$$

for all $b^m a^n$, $b^{m'} a^{n'}$ in S, $ba \neq 1$ and Z the ring of integers. We proved that this semigroup-ring coincides with the ring with identity element 1 generated by two elements a, b such that $ab = 1 \neq ba$. Let D be a ring with unity 1 and let $a, b \in D$ such that $ab = 1 \neq ba$. Let X be the sub-ring of D generated by a,b. Then we got the structure of ring X as the homomorphic image of the semigroup-ring $Z(N \times N)$ where $N \times N = \{(m, n) \mid m, n \in N\}$ is the semigroup with multiplication defined as

$$(m, n).(m', n') = \begin{cases} (m, n - m' + n') \text{ if } n \geqslant m', \\ (m' - n + m, n') \text{ if } n \leqslant m', \end{cases}$$

for all $(m, n), (m', n') \in N \times N$, with kernel not containing the element $(0, 0) - (1, 1)$ and studied some properties of such rings. The object of this paper is to investigate further the structure of a free semigroup-ring $R(S)$ where, R is an arbitrary ring with identity element 1. We prove that every element $x \in R(S)$ has a unique decomposition $x = y + z$, where $y = y_1 + y_2$, $y_1 \in R[a]$, $y_2 \in R[b]$ and $z \in R(S)(1 - ba)R(S) = P$. We prove that, P coincides with $P' = \{x \in R(S) \mid xb^n = 0$ for some $n \in N\}$ and $P'' = \{x \in R(S) \mid a^m x = 0$ for some $m \in N\}$. It is interesting to observe that the ideal P considered as a ring has structure of an infinite matrix ring over R with only a finite number of non-zero entries in each matrix. For any ring A we denote by $J(A)$ the Jacobson radical of A. We show that if R is an integral domain (ring without zero divisors) then P is a prime ideal properly containing $J(R(S))$. Moreover, we prove that $J(R(S)) \cong J(R)(S'')$, where $S'' = \{e_{ij} = b^i(1 - ba)a^j \in R(S) \mid i, j$ are non-negative integers$\}$, so that in particular $J(R(S)) = 0$

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if and only if $J(R) = 0$. Hence it follows that the free semigroup-ring $Z(S)$ is semi-simple in the sense of Jacobson (1964). We also prove that the only units of $\frac{R(S)}{P}$ are the products of units of R and powers of a or b, from which it follows that $\frac{R(S)}{P}$ is an integral domain but not a division ring and that $J\left(\frac{R(S)}{P}\right) = 0$. If besides R being an integral domain R is also left noetherian then so is $\frac{R(S)}{P}$, as a consequence of Goldie's theorem on noetherian prime rings we observe that $\frac{R(S)}{P}$ is left quotient simple. If R is a division ring we observe that $\frac{R(S)}{P} \cong F$ where, F is a left (right) Ore domain. In section 3, we study the structure of the free semigroup-ring $R(S')$ where, $S' = \{b^m a^m \mid m \in N\}$ is the subsemigroup of all the idempotents of S and R is an arbitrary ring.

§2. We shall first prove a few lemmas.

Lemma 2.1 — Any element of $R(S)$ reduces to a polynomial in $R[b]$ (in $R[a]$) by multiplying by a suitable power of b (of a) on the right (left).

PROOF : Any $x \in R(S)$ can be written in the form $x = \sum_{i=0}^{m} p_i(b)\, a^i$, where $p_i(b) \in R[b]$ for $i = 0, 1, \ldots, m$. Hence we have $xb^m = \sum_{i=0}^{m} p_i(b)\, b^{m-i}$. Likewise any $x \in R(S)$ can be reduced to an element of $R[a]$ by multiplying on the left by a suitable power of a.

Lemma 2.2 — If $P' = \{x \in R(S) \mid xb^m = 0$ for some $m \in N\}$ and $P'' = \{x \in R(S) \mid a^n x = 0$ for some $n \in N\}$, then P' and P'' are ideals in $R(S)$ and both contain $P = R(S)\,(1 - ba)\,R(S)$.

PROOF : P' is obviously a left ideal in $R(S)$. Now let $x \in P'$ so that $xb^n = 0$ for some $n \in N$ and let $y \in R(S)$. Reduce y to a polynomial in b as in lemma (2.1), say $yb^m \in R[b]$. Then $xyb^{m+n} = x(yb^m)\, b^n = (xb^n)\,(yb^m) = 0$. Hence $xy \in P'$. We observe that $(1 - ba)\, b = 0$ so that P is contained in P'. Analogous property of P'' can be verified in a similar way.

Let $P''' = \{x \in R(S) \mid x = p_1(a) + p_2(b), p_1(a) \in R[a], p_2(b) \in R[b]$, then we have the following lemma.

Lemma 2.3 — $P''' \cap P' = 0 = P''' \cap P''$.

PROOF : If $x = p(b) \in R[b]$, then $xb^n \neq 0 \;\forall\; n \in N$ unless $x = 0$. If $x = p_1(a)$ or $p_1(a) + p_2(b)$, where $0 \neq p_1(a) \in R[a], p_2(b) \in R[b]$ then $xb^n \in R[b]$

where n is the degree of $p_1(a)$ and hence $xb^{n+k} \neq 0 \ \forall\ k \in N$ unless $x = 0$, it follows that $P''' \cap P' = 0$. Likewise $P''' \cap P'' = 0$ can be verified.

Lemma 2.4 — If $P^{(iv)} = \{bxa - x \mid x \in R(S)\}$ then, $P^{(iv)} = P$.

Proof : Let $x = \sum_{m,n \in N} r(m, n)\, b^m a^n \in R(S)$, $r(m, n) \in R$. Then $bxa - x = \sum_{m,n \in N} r(m, n)\, (b^{m+1}a^{n+1} - b^m a^n) = \sum_{m,n \in N} r(m, n)\, b^m(1 - ba)\, a^n \in P$. Conversely any element of P has the form $\sum_{m,n \in N} r(m, n)\, b^m(1 - ba)\, a^n$ (since, $(ba - 1)\, b = 0 = a\,(ba - 1)$ by using the fact that $ab = 1 \neq ba) = bxa - x$
where, $x = -\sum_{m,n \in N} r(m, n)\, b^m a^n \in R(S)$.

Theorem 2.1 — P is isomorphic to an infinite matrix ring over R with only a finite number of non-zero entries in each matrix.

Proof : With each $x = \sum_{m,n \in N} r(m, n)\, b^m(1 - ba)\, a^n \in P$ associate the matrix $(r(m, n))$ of its coefficients in R which is unique since $R(S)$ is free. It follows that the mapping $x \to (r(m, n))$ is a $(1-1)$ onto map of P to the ring of infinite matrices on R with each matrix having only a finite number of nonzero entries. If

$$y = \sum_{l,k \in N} r(l, k)\, b^l(1 - ba)\, a^k \text{ then clearly } x + y \to (r(m, n)) + (r(l, k))$$

and

$$xy = \sum_{m,n,l,k \in N} r(m, n)\, s(l, k) b^m(1 - ba)\, a^n b^l(1 - ba)\, a^k$$

$$= \sum_{m,k \in N,\ n=l} r(m, n)\, s(l, k)\, b^m(1 - ba)\, a^k$$

(since $(1 - ba)\, b = 0 = a(1 - ba)$ and $(1 - ba)^2 = (1 - ba)$)
corresponds to $(r(m, n))\,(s(l, k))$.

Remark : We observed (Fotedar 1973) that the elements

$$\sigma_{ij}(ba) = b^{i+1}a^{j+1} - b^i a^j \neq 0 \ (= e_{ij} \text{ say})\ i, j \in N \text{ of } \sigma_{ij}(Z(S))$$

behave like matrix units. It is clear that P is precisely the semigroup-ring $R(S'')$ where $S'' = \{e_{ij} \mid i, j \in N\}$. This gives an alternative way of looking at P as an infinite matrix ring over R with a finite number of nonzero entries.

Theorem 2.2 — Any $x \in R(S)$ has a unique representation of the form $x = y + z$, $y \in P'''$, $z \in P$.

Proof : If $x = \sum_{m,n \in N} r(m, n)\, b^m a^n$, $r(m, n) \in R$ then by observing that

$$b^m a^n = (b^m a^n - b^{m-1}a^{n-1}) + (b^{m-1}a^{n-1} - b^{m-2}a^{n-2}) + \ldots + a^{n-m}$$

for $m \leqslant n$ and $b^m a^n = (b^m a^n - b^{m-1}a^{n-1}) + (b^{m-1}a^{n-1} - b^{m-2}a^{n-2}) + \ldots + b^{m-n}$ for $n \leqslant m$ the last terms being in P''' and rest of the terms in P by Lemma 2.4. Hence we have that

$$x = \sum_{m \geqslant n} r(m, n) \{(b^m a^n - b^{m-1}a^{n-1}) + (b^{m-1}a^{n-1} - b^{m-2}a^{n-2}) + \ldots + b^{m-n}\}$$
$$+ \sum_{m \leqslant n} r(m, n) \{(b^m a^n - b^{m-1}a^{n-1}) + (b^{m-1}a^{n-1} - b^{m-2}a^{n-2}) + \ldots a^{n-m}\} = z+y$$

where

$$z = \sum_{m \geqslant n} r(m, n) \{(b^m a^n - b^{m-1}a^{n-1}) + \ldots + (b^{m-(n-1)}\, a - b^{m-n})\}$$
$$+ \sum_{m \leqslant n} r(m, n) \{(b^m a^n - b^{m-1}a^{n-1}) + \ldots (ba^{n-(m-1)} - a^{n-m})\}$$

and

$$y = \sum_{m \geqslant n} r(m, n)\, b^{m-n} + \sum_{m<n} r(m, n)\, a^{n-m} \in P'''.$$

This representation is unique since $x = y + z = y' + z'$; $y, y' \in P'''$; $z, z' \in P$ implies $y - y' = z' - z \in P''' \cap P \subseteq P'' \cap P''' = 0$ by lemmas (2.2) and (2.3).

Theorem 2.3 — $P = P' = P''$

PROOF : To prove $P = P'$ we have in view of Lemma 2.2 only to show that $P \supseteq P'$. Let $x \in P'$ and let $x = y + z$, $y \in P'''$, $z \in P \subseteq P'$ be the decomposition of x given by Theorem 2.1. Then $y = x - z \in P''' \cap P' = 0$ by Lemma 2.3, so that $x = z \in P$. A similar argument shows that $P'' \subseteq P$.

Theorem 2.4 — Let R be an integral domain then,

(i) $\frac{R(S)}{P}$ is an integral domain and $\frac{R(S)}{P}$ is left (right) noetherian if R is left (right) noetherian.

(ii) $J(R(S)) \cong J(R)\,(S'')$ (S'' considered as a semigroup and the zero of S'' identified with the zero of $J(R)$ so that, in particular, here $J(R(S)) = J(R)(S'')$ and consequently $R(S)$ is semisimple if and only if R is semisimple and also $J(R(S)) \subset P$.

(iii) The only units of $\frac{R(S)}{P}$ are the products of units of R and powers of a or b.

(iv) $J\left(\frac{R(S)}{P}\right) = 0.$

PROOF : (i) We observe that $\frac{R(S)}{P} \cong R[a, a^{-1}]$, the infinite cyclic groupring over R. Then it is easy to see that $R[a, a^{-1}]$ is left noetherian if R is left noetherian by the Hilbert basis theorem (Lambek 1966, p. 70).

(ii) Let $x = \sum_{i,j \in N} r(i,j)\, b^i(1-ba)\, a^j \in J(R(S))$ and let $\max \{i \mid r(i,j) \neq 0\} = m$ and $\max \{j \mid r(m,j) \neq 0\} = n$. Then $a^m x b^n = r(m,n)(1-ba) \in J(R(S))$ so that $r(m,n) \neq 1$. Let $y = \sum_{l,k \in N} s(l,k)\, b^l(1-ba)a^k$ be its quasi-inverse so that

$$(1 - r(m,n)(1-ba))(1 - \sum_{l,k \in N} s(l,k)\, b^l(1-ba)\, a^k) = 1$$

$$= (1 - \sum_{l,k \in N} s(l,k)\, b^l(1-ba)\, a^k) \,.\, (1 - r(m,n)(1-ba)).$$

It is easily seen that $l \ngtr 0$ and $k \ngtr 0$ so that we have

$$(1 - r(m,n)(1-ba))(1 - s(0,0)(1-ba)) = 1$$

$$= (1 - s(0,0)(1-ba))(1 - r(m,n)(1-ba))$$

or that

$$(r(m,n) \circ s(0,0))(1-ba) = 0 = (s(0,0) \circ r(m,n))(1-ba)$$

which implies that

$$s(0,0) \circ r(m,n) = 0 = r(m,n) \circ s(0,0) = 0.$$

Hence $r(m,n)$ is quasi-regular $(q.r.)$. Since

$$r(m,n)(1-ba) \in J(R(S)),\ r(m,n)\, s(1-ba) \in J(R(S))$$

for all $s \in R$ it follows, as above, that $r(m,n)s$ is $q.r.$ Hence $r(m,n) \in J(R)$.

Let $n' = \max \left\{ \{j \mid r(m,n) \neq 0\} \setminus \{n\} \right\}$ if it exists.

Then $a^m x b^{n'} = r(m,n)(1-ba)\, a^{n-n'} + r(m,n')(1-ba) \in J(R(S))$. But $r(m,n)(1-ba)\, a^{n-n'} \in J(R(S))$ so that $r(m,n)(1-ba) \in J(R(S))$ which by a similar argument as above implies that $r(m,n')$ is in $J(R)$. By repeating this type of argument we can show that all the coefficients $r(m,j) \in J(R)$. Next choose

$$m_1 = \max \left\{ \{i \mid r(i,j) \neq 0\} \setminus \{m\} \right\}$$

and

$$n = \max \{j \mid r(m_1, j) \neq 0\}$$

then

$$a^{m_1} x b^{n_1} = \sum_{j \geqslant n_1} r(m,j)\, b^{m-m_1}(1-ba)\, a^{j-n_1} + r(m_1,n_1)(1-ba).$$

But we have already shown that

$$\sum_{j \geqslant n_1} r(m,j)\, b^{m-m_1}(1-ba)\, a^{j-n_1} \in J(R(S))$$

hence

$$r(m_1,n_1)(1-ba) \in J(R(S)),$$

so that $r(m_1, n_1) \in J(R)$. Repeating this argument we can show that

$$r(i, j) \in J(R) \ \forall\ i, j \in N$$

from which it follows that $x \in J(R)(S'')$. In particular it follows that if $J(R(S)) \neq 0$ then $J(R) \neq 0$. Conversely let $0 \neq r \in J(R)$ we show that $0 \neq r(1 - ba)\, R(S)$ is a right quasi-regular right ideal. Any element of $r(1 - ba)\, R(S)$ has the form $r(1 - ba)\, p(a)$ where, $p(a) \in R[a]$. Let r_0 be the constant term of $p(a)$ such that $p(a) = p_0(a) + r_0$. Since $r \in J(R)$ it follows that $rr_0 \in J(R)$ so that $\exists$ an $s_0 \in J(R)$ such that

$$rr_0 \circ s_0 = s_0 \circ rr_0 = 0.$$

We now observe that

$$r(1 - ba)\, p(a) \circ (1 - ba)\, \{ -rp_0(a) + s_0 rp_0(a) + s_0\} = 0,$$

so that

$$r(1 - ba) \in r(1 - ba)\, R(S) \subseteq J(R(S)).$$

It follows that if $J(R) = 0$ then $J(R(S)) = 0$. We now show that any element of the form $rb^n(1 - ba)\, p(a)$, $n > 0$, $p(a) \in R[a]$ and $r \in J(R)$ is right quasi-regular. Let the degree of $p(a)$ be equal to m. If $m < n$, then $rb^n(1 - ba)\, p(a) \circ (-rb^n(1 - ba)\, P(a)) = 0$. If $m \geqslant n$, then let the coefficient of a^n is $p(a)$ be r_n and let $rr_n \circ s_n = 0$ for some $s_n \in J(R)$. Then we have

$$rb^n(1 - ba)\, p(a) \circ b^n(1 - ba)\, (s_n - 1)\, rp(a) = 0.$$

$$\text{Let } x = \sum_{i,j \in N} r(i, j)\, b^i(1 - ba)\, a^j = \sum_{l \in N} r_l b^l(1 - ba)\, p_l(a) \in J(R)\,(S''),$$

and let $\min\{l \mid r_l \neq 0\} = m_1$. Then we have

$$x \circ b^{m_1}(1 - ba)\, (s_{m_1} - 1)\, r_{m_1} p_{m_1}(a) = \sum_{l \neq m_1} r_l b^l(1 - ba)\, p_l(a)$$

$$- \sum_{l \neq m_1} r_l b^l(1 - ba)\, p_l(a) \,.\, b^{m_1}(1 - ba)\, (s_{m_1} - 1)\, r_{m_1} p_{m_1}(a)$$

$$= \sum_{l \neq m_1} u_l b^l(1 - ba)\, p_l'(a),\ u_l \in J(R), \text{ (say)}$$

where s_{m_1} is given by $r_{m_1} t_{m_1} \circ s_{m_1} = 0$, t_{m_1} being the coefficient of a^{m_1} in $p_{m_1}(a)$. By repeating the above process for each $l \in N$ we get the right quasi-inverse of x. For any $y \in R(S)$, xy has the same form as x, hence $x \in J(R(S))$. Since $J(R)\,(S'') \subseteq P$, it follows that $J(R(S)) \subseteq P$. Since $(1 - ba)^2 = (1 - ba) \in P$, it is clear that $J(R(S)) \subset P$. In general for any

$$x = \sum_{m,n \in N} r(m, n)\, (b^m a^n - b^{m+1} a^{n+1}) \in J(R(S)),\ r(m, n) \in J(R),$$

the mapping $x \to \sum_{m,n \in N} r(m, n)\, e_{mn}$ is an isomorphism of $J(R(S))$ upon $J(R)(S'')$.

(iii) Since $\frac{R(S)}{P} \cong R[a, a^{-1}]$ the proof of this statement is identical to finding units in commutative integral group-rings given by Sehgal (1970).

(iv) Let $x \in J\left(\frac{R(S)}{P}\right)$ so that $1 - x = ra^k$, where r is a unit in R and $k \gtrless 0$. If $k \geqslant 0$ then $(1 - xa) = 1 - a + ra^{k+1}$ which is not a unit unless $r = 1, k = 0$ i.e., unless $x = 0$. Similarly if $k < 0$ then $1 - xb = 1 - b + rb^{k+1}$ is not a unit unless $x = 0$. Hence $J\left(\frac{R(S)}{P}\right) = 0$.

Corollary 1 — The free semigroup-ring $Z(S)$ is semisimple.

Corollary 2 — If R is a division ring, then $\frac{R(S)}{P}$ is a semisimple principal left (right) ideal domain.

Corollary 3 — If R is a left (right) noetherian integral domain then, $\frac{R(S)}{P}$ is left (right) quotient simple.

For proof see Goldie (1958).

Corollary 4 — If R is a division ring, then $\frac{R(S)}{P} \cong F$, where F is a left or right Ore domain.

For proof see Goldie (1962).

§3. Let $S' = \{\epsilon_m = b^m a^m \in S \mid m \in N\}$ be the set of all the idempotents of S. Then S' is a commutative subsemigroup with unity $\epsilon_0 = 1$ and the multiplication in S' reduces to $\epsilon_m \cdot \epsilon_n = \epsilon_{max\,(m,\,n)}$. Let $R(S')$ denote the free semigroup-ring of S' over an arbitrary ring R. For $\alpha = \sum_{i \in N} r_i \epsilon_i \in R(S')$, let $\alpha_n = \sum_{i=0}^{n} r_i \epsilon_i$ for each $n \in N$ and $|\alpha| = \sum_{i \in N} r_i$. Denote the mapping $\alpha \to \alpha_n$ of $R(S')$ into itself by ϕ_n and the mapping $\alpha \to |\alpha|$ of $R(S')$ into R by ψ. Then clearly ψ is an epimorphism. Further denote the restriction of ψ to $\phi_n(R(S'))$ by ψ_n. Then we have

Lemma (3.1) — ϕ_n is an endomorphism and ψ_n is an epimorphism.

PROOF : Let $\alpha = \sum_{i \in 1}^{n} r_i \epsilon_i, = \sum_{j \in N} s_j \epsilon_j \in R(S')$. Then clearly $(\alpha + \beta)_n = \alpha_n + \beta_n$ and $(\alpha\beta)_n = \sum_{\substack{i,j \\ \max(i,j) \leqslant n}} r_i s_j \epsilon_i \epsilon_j = \sum_{i=0}^{n} r_i \epsilon_i \sum_{j=0}^{n} s_j \epsilon_j = \alpha_n \beta_n$. It follows in particular that $\phi_n(R(S'))$ is a subring of $R(S')$ and hence also that ψ_n is a homomorphism which is obviously onto since, $\psi_n(r\epsilon_n) = r \;\forall\; r \in R$.

Corollary — $(\alpha \circ \beta)_n = \alpha_n \circ \beta_n$ and $|(\alpha \circ \beta)_n| = |\alpha_n| \circ |\beta_n|$.

Let R_∞ be the ring of all infinite sequences of elements of R such that each sequence after a finite stage becomes stationary i.e., the elements of R have the form $(r_0, r_1, r_2, \ldots, r_n, r_n, \ldots)$ for $n < \infty$, and addition and multiplication are defined coordinatewise.

Lemma 3.2 — $R(S') \cong R_\infty$ under the mapping

$$\alpha \to (|\alpha_0|, |\alpha_1|, \ldots, |\alpha_n|, |\alpha_n|, \ldots)$$

where

$$\alpha = \sum_{i=0}^{n} r_i \epsilon_i.$$

Proof : From Lemma 3.1, it follows that the mapping is indeed a homomorphism. Clearly $|\alpha_n| = 0, \forall\, n \in N \Leftrightarrow \alpha = 0$ and the element

$$(s_0, s_1, \ldots s_m, s_m, \ldots) \in R_\infty$$

is the image of

$$s_0 \epsilon_0 + (s_1 - s_0)\epsilon_1 + (s_2 - s_1)\epsilon_2 + \ldots + (s_m - s_{m-1})\epsilon_m \in R(S').$$

Denote by $J(R(S'))$ and $J(R)$ the Jacobson radical of $R(S')$ and R respectively. Then we have

Theorem 3.1 — $J(R(S')) = \{\sum_{i \in N} r_i \epsilon_i \in R(S') \mid r_i \in J(R)\ \forall\ i \in N\}$,

so that in particular $R(S')$ is semisimple or a radical ring iff R is semisimple or a radical ring respectively.

Proof : We observe by Lemma 3.2 that

$$\alpha = \sum_{i=0}^{n} r_i \epsilon_i \in J(R(S')) \Leftrightarrow (|\alpha_0|, |\alpha_1|, \ldots, |\alpha_n|, |\alpha_n|, \ldots)$$

$$\in J(R_\infty) \Leftrightarrow |\alpha_i| \in J(R)\ \forall\ i = 0, 1, \ldots, n.$$

i.e., $r_0, r_0 + r_1, \ldots, r_0 + r_1 + \ldots + r_n \in J(R) \Leftrightarrow r_i \in J(R), i = 0, 1, 2, \ldots, n.$

Denote by R^∞ the infinite discrete direct sum of copies of R and let K denote the kernel of ψ. Clearly R^∞ is an ideal of R_∞. Let i_1 denote the injection of K into $R(S')$, i_2 the injection of R^∞ into R_∞ and let $\pi : (r_1, r_2, \ldots, r_m, r_m, \ldots) \to r_m$ be a map of R_∞ into R. Then clearly π is an epimorphism and

$$\ker(\pi) = \{(r_0, r_1, \ldots, r_n, 0, 0, \ldots) \in R_\infty \mid n \in N\} = R^\infty$$

and $R^\infty \cong K$ by Lemma 3.2. Thus we have

Theorem 3.2 —

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i_1} & R(S') & \xrightarrow{\psi} & R & \longrightarrow & 0 \\ & & \updownarrow \text{iso.} & & \updownarrow \text{iso.} & & \updownarrow \text{Id map} & & \\ 0 & \longrightarrow & R^{\infty} & \xrightarrow{i_2} & R_{\infty} & \xrightarrow{\pi} & R & \longrightarrow & 0 \end{array}$$

is a commutative diagram of rings.

We remark that R^{∞} is neither noetherian nor artinian it follows that K and hence $R(S')$ cannot be noetherian or artinian. It also follows from the theorem that the problem of finding the ideals of $R(S')$ is reduced to that of finding the ideals of R_{∞}, which we proceed to do. Let R have unity element 1 and let I be a right ideal of R_{∞}. Let I_i be the ideal formed by the ith coordinates of the elements of I, for $i = 0$, 1, ... hr. Further let M be the set of all the stationary coordinates of elements of I so that

$$M = \{r_n \in R \mid (r_0, r_1, \ldots, r_n, r_n, \ldots) \in I, n \geqslant 0\}.$$

Then clearly M is also an ideal in R. Let

$$J_i = \bigcap_{k=i}^{\infty} I_k, \; i = 0, 1, 2, \ldots$$

and

$$J = \text{lim. inf. } I_k = \bigcup_{i=0}^{\infty} J_i.$$

Then it is easily seen that $M \subseteq J$ and that if

$$I' = \{(a_0, a_1, \ldots, a_{i-1}, a, a, \ldots) \mid a_j \in I_j, j = 0, 1, \ldots, i-1,$$
$$a \in M \cap J_i, i = 0, 1, \ldots\}$$

then $I' = I$ since $(a_0, 0, 0, \ldots), (0, a_1, 0, 0, \ldots), \ldots (0, 0, \ldots, 0, a_i, a_i, \ldots) \in I$.

Conversely, if $I_k, k = 0, 1, \ldots$ is a given sequence of ideals of R and $M \subseteq J =$ lim. inf. I_k is an ideal of R then the set I' constructed as above is an ideal of R_{∞}.

Theorem 3.3 — (i) If R is an integral domain then the set of all right divisors of zero of $R(S')$ is $K_1 = \{\alpha \in R(S') \mid \mid \alpha_n \mid = 0$ for some $n \in N\}$.

(ii) If R has unity then the set of all right units of $R(S')$ is $K_2 = \{\alpha \in R(S') \mid \mid \alpha_n \mid$ is a right unit in $R \; \forall \; n \in N\}$.

(iii) $R(S')$ is von-Neumann regular iff R is von-Neumann regular.

PROOF : It follows from Lemmas 3.1 and 3.2.

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