INTENSITY OF A NULL POINT SOURCE IN A STATIONARY SPHERICALLY SYMMETRIC SPACE-TIME

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Intensity of an electromagnetic wave, emitted by a point source falls off as the square of the distance in flat space-time (i.e., Euclidean space). No effort has been made so far to find its behaviour, in general, for a curved space-time. Though some efforts have been made in the case of Schwarzschild and Robertson-Walker space-time, however, nothing is done to find its behaviour in general for a space-time having a certain type of symmetry. In the present paper we calculate the intensity of null wave (i.e., electromagnetic wave) emitted by a point source in a stationary spherically symmetric space-time. The developed intensity formula in this paper reduces to its usual values in the cases of flat and Schwarzschild space-times.

§1. Intensity of a point source emiting null waves (i.e., electromagnetic or light wave) is given by (Kristian and Sachs 1966)

$$I = \frac{L\delta\Omega}{4\pi A_0(1 + z)^2} \qquad ...(1.1)$$

where L is the intrinsic luminosity of the source, $\delta\Omega$ the solid angle of the light beam at the point source, z is the red shift from source to observer, and A_0 is the area of the beam at the observer. In a flat space the wave fronts, for a point source, are concentric spherical shells and therefore the area A_0 of a beam of light varies as the square of the radius d of spherical shell (the distance between the source and the observer). The intensity, for an observer at rest relative to the point source, therefore in flat space-time is given by

$$I = \frac{L}{4\pi d^2}.$$

To find the intensity in a curved space-time, A_0 in eqn. (1.1) must be calculated for that space-time. This is done by examining, how the null waves expand in that space-time for a point source. The purpose of this paper is to study the expansion of null waves, emitted by a point source in a stationary spherically symmetric space time and to calculate A_0. Intensity then, is immediate from (1.1)

§2. The area A, in equation (1.1) is related to the expansion θ of the light beam by (Sachs 1961)

$$A'/A = 2\theta \qquad \text{...(2.1)}$$

where ($'$) is the differentiation with respect to the central ray's affine parameter. The expansion θ is, however, related to the principal curvatures of the wave front $K_\pm$ by Kantowski (1968)

$$\theta = \frac{K_+ + K_-}{2}. \qquad \text{...(2.2)}$$

If we rewrite the principal curvatures in terms of the major and minor dimensions of the wave front, $D_\pm$ (Dwivedi 1970)

$$K_\pm = \frac{D'_\pm}{D_\pm}. \qquad \text{...(2.3)}$$

Putting (2.2) and (2.3) in (2.1) we get

$$A'/A = \frac{[D'_+ D_-]}{D_+ D_-}. \qquad \text{...(2.4)}$$

The solution of eqn. (2.4) when evaluated at the observer gives us

$$A_0 = [D_+ D_-]_0.$$

To calculate D_+ and D_- we must go directly to the case of a stationary spherically symmetric space-time which is given by the Riemannian metric (Bergmann 1950)

$$ds^2 = g_1(r)\, dt^2 - g_2(r)\, dr^2 - r^2\, [d\theta^2 + \sin^2\theta\, d\phi^2] \qquad \text{...(2.5)}$$

where r, t, θ, ϕ have their usual meanings. Consider now the motion of a light beam in a space-time given by the metric (2.5). Using the variational principle and the Euler-Lagrange equations, the calculation of the equations of motion of null geodesics (i.e., light rays) are straightforward, and we get the tangents to the null geodesics making up the null hypersurface $x^a(\lambda, \beta, l)$

$$k^t = \frac{dt}{d\lambda} = k/g_1 \qquad \text{...(2.6)}$$

$$k^r = \frac{dr}{d\lambda} = k\left[\frac{1}{g_1 g_2} - \frac{l^2}{r^2 g_2}\right]^{1/2} \qquad \text{...(2.7)}$$

$$k^\theta = \frac{d\theta}{d\lambda} = \frac{kl}{r^2} \cos\Phi \sin\beta \qquad \text{...(2.8)}$$

$$k^\phi = \frac{d\phi}{d\lambda} = \frac{kl}{r^2} \cos\beta/\sin^2\theta \qquad \text{...(2.9)}$$

where λ is the affine parameter, β the isotropy parameter, and l the impact parameter (Dwivedi 1970). Vector k^a defind above is normal to the null hyper surface $x^a(\lambda, \beta, l)$ and tangent to the null geodesic. Let us define $e^a_\pm$ as

$$e^a_+ = \frac{\partial x^a}{\partial l}(\lambda, \beta, l) = \frac{\partial}{\partial l}\int k^a d\lambda = \frac{\partial}{\partial l}\left[\int_R^r \frac{k^a}{k^r}\,dr\right] \quad \text{...(2.10)}$$

$$e^a_- = \frac{\partial x^a}{\partial \beta}(\lambda, \beta, l) = \frac{\partial}{\partial \beta}\int k^a d\lambda = \frac{\partial}{\partial \beta}\left[\int_R^r \frac{k^a}{k^r}\,dr\right] \quad \text{...(2.11)}$$

where $r = R$, $\phi = 0$, $\theta = 2\pi$ is the position of the point source. We now show that $D_\pm$ defined in (2.3) are actually the magnitudes of vectors $e^a_\pm$ defined in (2.10) and (2.11).

Let m^a be a vector such that $m^a k_a = 1$, $m^a m_a = m_a e^a_\pm = 0$. Note also from equations (2.10), (2.11), and (2.6) to (2.9)

$$k^a k_a = k_a e^a_\pm = 0, \quad \text{...(2.12)}$$

$$e^a_- e^b_- g_{ab} = |e_-|^2 = r^2[\sin^2\phi + \cos^2\phi\cos^2\theta] \quad \text{...(2.13)}$$

$$e^a_+ e^b_+ g_{ab} = |e_+|^2 = r^2[r]\int_R^r \frac{dr}{r^2[r]^{3/2}} \quad \text{...(2.14)}$$

where g_{ab} is the Riemann metric given by (2.5), $|e_+|$ and $|e_-|$ the magnitudes of vectors $e^a_\pm$, and $[r] = \left[\frac{1}{g_1 g_2} - \frac{l^2}{r^2 g_2}\right]$.

Using the fact

$$\frac{\partial^2 x^a}{\partial l \partial \lambda} = \frac{\partial^2 x^a}{\partial \lambda \partial l},$$

$$\frac{\partial^2 x^a}{\partial \beta \partial \lambda} = \frac{\partial^2 x^a}{\partial \lambda \partial \beta},$$

we get

$$k^a{}_{;b}\, e^b_\pm = e^a_{\pm;b}\, k^b \quad \text{...(2.15)}$$

where (;) denotes the usual covariant differentiation. Writing $e^a_{\pm;b} k^b$ in terms of the vectors k^a, m^a, and $e^a_\pm$ we get

$$k^a{}_{;b}\, e^b_{\pm} = e^a_{\pm\,;b}\, k^b = A_{\pm} k^a + B_{\pm} m^a + C_{\pm} e^a_{+} + F_{\pm} e^a_{-}. \qquad ...(2.16)$$

Transvecting (2.16) with k^a and using (2.12) and the fact $k^a k_{a;b} = 0$ we obtain

$$B_{\pm} = 0. \qquad ...(2.17)$$

Similarly transvecting the two equations in (2.16) with e^a_{-} and e^a_{+}

$$e^a_{-}\, k_{a;b}\, e^b_{+} = e^a_{-}\, e_{+a;b}\, k^b$$

$$e^a_{+}\, k_{a;b}\, e^b_{-} = e^a_{+}\, e_{-a;b}\, k^b.$$

Adding the above two equations and using (2.12)

$$e^a_{-}\, k_{a;b}\, e^b_{+} + e^a_{+}\, k_{a;b}\, e^b_{-} = [e^a_{-}\, e_{+a}]_{;b}\, k^b = 0.$$

Since k^a is normal to the null hypersurface (i.e., $k_{a;b} = k_{b;a}$)

$$e^a_{-}\, k_{a;b}\, e^b_{+} = e^a_{+}\, k_{a;b}\, e^b_{-} = 0 \qquad ...(2.18)$$

Equation (2.18) and (2.16) imply

$$C_{-} = F_{+} = 0 \qquad ...(2.19)$$

Putting (2.17) and (2.19) in (2.16) we obtain

$$k^a{}_{;b} e^b_{\pm} = e^a_{\pm\,;b}\, k^b = \chi_{\pm}\, [\sigma_{\pm} k^a + e^a_{\pm}] \qquad ...(2.20)$$

where we have put $\chi_{+} = C_{+}$, $\chi_{-} = F_{-}$, and $\sigma_{\pm} = A_{\pm}/\chi_{\pm}$. Since k^a is tangent to null geodesic (i.e., $k^a{}_{;b}\, k^b = 0$) eqn. (2.20) can be written as

$$k^a{}_{;b}\, [\sigma_{\pm} k^b + e^b_{\pm}] = \chi_{\pm}\, [\sigma_{\pm} k^a + e^a_{\pm}]. \qquad ...(2.21)$$

If we let $\eta^a_{\pm} = \sigma_{\pm} k^a + e^a_{\pm}$

$$k_{a;b}\, \eta^b_{\pm} = \chi_{\pm} \eta_{\pm a}. \qquad ...(2.22)$$

This says that $\eta^a_{\pm}$ are the eigen-direction of $k_{a;b}$ with eigenvalue $\chi_{\pm}$, i.e., $\eta^a_{\pm}$ are principal curvature directions with principal curvatures $\chi_{\pm}$. The values of $\chi_{\pm}$ are immediate from (2.20)

$$\chi_{\pm} = \frac{{}_{+}e^a_{\pm}\, e_{\pm a;b} k^b}{|\, e_{\pm}\, |^2} = \frac{|\, \acute{e}_{\pm}\, |}{|\, e_{\pm}\, |} \qquad ...(2.23)$$

The major and minor dimensions of the wave front defined in (2.3) are, therefore given by

$$D_+ = | e_+ | \, \delta l = \delta l \, r \, [r]^{1/2} \int_R^r \frac{dr}{r^2 \, [r]^{3/2}} \qquad \text{...(2.24)}$$

$$D_- = | e_- | \, \delta\beta = \delta\beta \, r \sin \phi \qquad \text{...(2.25)}$$

where we have oriented the coordinates so that the central ray is in $\theta = \pi/2$ plane. To get the intensity of a point source we put both $D_\pm$ into eqn. (1.1) along with the solid angle (Dwivedi 1970).

$$\delta\Omega = \frac{l g_1 \delta l \delta\beta}{R^2 \left[1 - \frac{l^2}{R^2} g_1 \right]^{1/2}}. \qquad \text{...(2.26)}$$

The intensity I of a point source is then given by

$$I = \frac{l L g_1 / R^2 (1 + z)^2}{4\pi r^2 \sin \phi \left[1 - \frac{l^2}{R^2} g_1 \right]^{1/2} [r]^{1/2} \int_R^r \frac{dr}{r^2 \, [r]^{1/2}}}. \qquad \text{...(2.27)}$$

This is the intensity of a point source as seen by an observer with coordinates r, t, ϕ. in a stationary spherically symmetric space-time. I in equation (2 27) reduces to (1.2) in a flat space-time, i.e., if we put $g_1 = g_2 = 1$ in (2.27) we obtain $I \propto 1/r^2$. It is also straightforward to calculate I in case of a schwarzschild space-time by simply putting $g_1 = 1/g_2 = (1 - (2m/r))$. The intensity thus obtained for Schwarzschild case is in complete agreement with the results of Dwivedi (1970).

Before closing we should mention an interesting point about the intensity of a point source in a stationary spherically symmetric space-time. Note the term $\sin \phi$ in the denominator of (2.27) where ϕ is given by (2.9)

$$\phi = kl \int_R^r \frac{dr}{r^2 \left[\frac{1}{g_1 g_2} - \frac{l^2}{r^2 g_2} \right]^{1/2}}.$$

In a flat space $\sin \phi = 0$ only at the source as can be seen by putting $g_1 = g_2 = 1$. Therefore refocusing of light rays is not possible in flat space-time. However in a curved space-time if the gravitational field is strong enough ϕ in the above equation can have the values π, 2π, etc. making $\sin \phi = 0$ at some points in the space other than the source itself. Therefore in a curved space-time we could see images of a point source.

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