A NOTE ON PADÉ APPROXIMATIONS OF ORDER (m, n) TO THE HYPERGEOMETRIC FUNCTION $F(\alpha, \beta, \gamma; x)$

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It is wellknown that a wide class of special functions including the elliptic integrals can be derived from the hypergeometric function $F(\alpha, \beta, \gamma; x)$ by suitable choices of the values of the three parameters. We have sought to approximate $F(\alpha, \beta, \gamma; x)$ by a rational function of order (n, n). The Padé tables for $n = 6$ for two sets of values of α, β, and γ have been computed using a recursive scheme suggested by Longman (1971). The results are presented in this paper. They are in good agreement with the values obtained directly from the corresponding hypergeometric series. Further computations of higher order Padé tables are in progress.

INTRODUCTION

The purpose of this note is to derive the linear system of equations which determine the coefficients of the rational Padé approximation of order (m, n) to the hypergeometric function $F(\alpha, \beta, \gamma; x)$ and solve them numerically for a set of values of α, β, and γ by a recursion scheme suggested by Longman (1971). The method of Longman avoids the matrix inversion procedure altogether. A brief outline of this scheme as applied to our problem is given in the Section 3. The purpose of this insertion is to make the article self-contained.

It is well known (Erdélyi 1953) that a wide class of special functions including the elliptic integrals which are often used in applied mathematics can be derived from the hypergeometric function by suitable choices of the values of the three parameters α, β, and γ. Hence it is of interest to investigate the rational Padé approximation of order (m, n) to $F(\alpha, \beta, \gamma; x)$. The approximations for the three kinds of elliptic integrals based on the Padé approximations of the square root have been discussed recently (Luke 1970). Padé approximation is relatively easy to obtain and can be used to derive the appropriate continued fractions so useful in high-speed computations (Fike 1968). For an exhaustive treatment (theoretical as well as numerical) of the Padé and related approximations of special functions (including the hypergeometric function and its generalizations) based on rational and Chebyshev series expansions (Luke 1969). It appears, however, that the numerical treatment of eqns. (6) and (7) of Section 2 below by Longman's technique has not been given by any author. This is one of the important reasons for taking up the present investigation.

2. Derivation of the Equations Determining the Coefficients of the Rational Function

The hypergeometric function $F(\alpha, \beta, \gamma; x)$ can be expanded in an infinite power series of the form :

$$F(\alpha, \beta, \gamma; x) = 1 + \frac{\alpha.\beta}{1!\gamma} x + \frac{\alpha(\alpha+1)\,\beta(\beta+1)\,x^2}{2!\,\gamma(\gamma+1)} + \ldots \qquad \ldots(1)$$

about the origin which is a regular singular point of the hypergeometric differential equation. The series (1) converges for $|x| < 1$ for all finite values of α, β and for all finite values of γ excluding negative integer values (Ince 1956). The above assumptions regarding the variable x and the parameters α, β and γ are implicit in all subsequent discussions.

A Padé table for $F(\alpha, \beta, \gamma; x)$ is a matrix whose elements are distinct rational approximations of $F(\alpha, \beta, \gamma; x)$ of various orders. Such a table is called normal if no two positions are occupied by the same rational function. This is possible when the rational functions are irreducible (Perron 1950). We place at the intersection of the $(m+1)$th row and $(n+1)$th column of the Padé table a rational function

$$R_{mn}(x) = \frac{P_{mn}(x)}{Q_{mn}(x)} \qquad \ldots(2)$$

where

$$P_{mn}(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \qquad \ldots(3)$$

$$Q_{mn}(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m \qquad \ldots(4)$$

with the property that the power series expansions of $R_{mn}(x)$ agrees with $F(\alpha, \beta, \gamma; x)$ as far as the term in x^{m+n}. Thus in a normal Padé table,

$$\left(1 + \frac{\alpha.\beta}{1!\gamma} x + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)} x^2 + \ldots\right)(b_0 + b_1x + \ldots + b_mx^m)$$
$$= a_0 + a_1x + \ldots + a_nx^n + O.x^{n+1} + \ldots + O.x^{m+n} + O.x^{m+n+1}. \qquad \ldots(5)$$

Equation (5) is an identity and hence comparing the coefficients of like powers of x on both sides we get the required equations to determine the coefficients $a_0, a_1, \ldots a_n$, $b_0, b_1, \ldots b_m$ of the rational approximant $R_{mn}(x)$. These are

$$\left.\begin{aligned}
&b_0 = a_0\\
&\frac{\alpha.\beta}{1!\gamma} b_0 + b_1 = a_1\\
&\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)} b_0 + \frac{\alpha.\beta}{1!\gamma} b_1 + b_2 = a_2\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
&\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\,\beta(\beta+1)\ldots(\beta+n-1)}{n!\,\gamma(\gamma+1)\ldots(\gamma+n-1)} b_0\\
&+ \frac{\alpha(\alpha+1)\ldots(\alpha+n-2)\,\beta(\beta+1)\ldots(\beta+n-2)}{(n-1)!\,\gamma(\gamma+1)\ldots(\gamma+n-2)} b_1 +
\end{aligned}\right\} (n+1) \text{ eqns.}$$

(*equation continued on p.* 629)

$$+\frac{\alpha(\alpha+1)\ldots(\alpha+n-m-1)\,\beta(\beta+1)\ldots(\beta+n-m-1)}{(n-m)!\,\gamma(\gamma+1)\ldots(\gamma+n-m-1)}\,b_m = a_n \qquad \ldots(6)$$

$$\frac{\alpha(\alpha+1)\ldots(\alpha+n)\,\beta(\beta+1)\ldots(\beta+n)}{(n+1)!\,\gamma(\gamma+1)\ldots(\gamma+n)}\,b_0$$

$$+\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\,\beta(\beta+1)\ldots(\beta+n-1)}{n!\,\gamma(\gamma+1)\ldots(\gamma+n-1)}\,b_1$$

$$+\ldots+\frac{\alpha(\alpha+1)\ldots(\alpha+n-m)\,\beta(\beta+1)\ldots(\beta+n-m)}{(n-m+1)!\,\gamma(\gamma+1)\ldots(\gamma+n-m)}\,b_m = 0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{\alpha(\alpha+1)\ldots(\alpha+n+m-1)\,\beta(\beta+1)\ldots(\beta+n+m-1)}{(n+m)!\,\gamma(\gamma+1)\ldots(\gamma+n+m-1)}\,b_0$$

$$+\frac{\alpha(\alpha+1)\ldots(\alpha+n+m-2)\,\beta(\beta+1)\ldots(\beta+n+m-2)}{(n+m-1)!\,\gamma(\gamma+1)\ldots(\gamma+n+m-2)}\,b_1$$

$$+\ldots+\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\,\beta(\beta+1)\ldots(\beta+n-1)}{n!\,\gamma(\gamma+1)\ldots(\gamma+n-1)}\,b_m = 0.$$

It is assumed that $n > m$. Adding to (6) the normalizing condition

$$b_0 = 1. \qquad \ldots(7)$$

We have $(m + n + 2)$ equations to determine as many unknowns ($n + 1$ a's and $m + 1$ b's). Condition (7) holds good in what follows.

3. Recursive Scheme for Determining a_i's and b_i's

Equation (5) characterizes (m, n) problem, i.e., the problem of computing P_{mn} and Q_{mn}. Consider now the $(m, n - 1)$ and the $(m - 1, n)$ problems. Corresponding to eqn. (5) we have

$$\left(1+\frac{\alpha.\beta}{1!\,\gamma}x+\ldots\right)(b_0+b_1x+\ldots+b_mx^m)$$

$$= a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + O.x^n + \ldots + O.x^{m+n-1} + O(x^{m+n}) \qquad \ldots(8)$$

for the $(m, n - 1)$ problem and

$$\left(1+\frac{\alpha.\beta}{1!\,\gamma}x+\ldots\right)(b'_0+b'_1x+\ldots+b'_{m-1}x^{m-1})$$

$$= a'_0 + a'_1x + \ldots + a'_nx^n + O.x^{n+1} + O.x^{n+2} + \ldots + O.x^{n+m-1} + O(x^{m+n}) \qquad \ldots(9)$$

for the $(m-1, n)$ problem. Here the coefficients for the $(m-1, n)$ problem have been indicated by primes, while the $(m, n-1)$ coefficients have been left unprimed.

Subtracting (9) from (8) we deduce the relation

$$\left(1+\frac{\alpha.\beta}{1!\gamma}x+\dots\right)[(b_0-b_0')+(b_1-b_1')x+\dots+(b_{m-1}-b_{m-1}')x^{m-1}$$
$$+b_m x^m]$$
$$=(a_0-a_0')+(a_1-a_1')x+\dots+(a_{n-1}-a_{n-1}')x^{n-1}-a_n'x^n$$
$$+O.x^{n+1}+O.x^{n+2}+\dots+O.x^{m+n-1}+O(x^{m+n}) \quad \dots(10)$$

Now $b_0=b_0'=1$ by (7) and $a_0=a_0'=1$ by the first equations (6), and so dividing out by a factor x we have from (10)

$$\left(1+\frac{\alpha.\beta}{1!\gamma}x+\dots\right)[(b_1-b_1')+(b_2-b_2')x+\dots+(b_{m-1}-b_{m-1}')$$
$$\times x^{m-2}+b_m x^{m-1}]$$
$$=(a_1-a_1')+(a_2-a_2')x+\dots+(a_{n-1}-a_{n-1}')x^{n-2}-a_n'x^{n-1}$$
$$+O.x^n+O.x^{n+1}+O.x^{m+n-2}+O(x^{m+n-1}). \quad \dots(11)$$

In general $b_1-b_1'\neq 1$, but we obtain $(m-1, n-1)$ problem if we divide both sides of (11) by b_1-b_1' which cannot be zero in a normal Padé table.

Using asterisks to denote coefficients in the solution of the $(m-1, n-1)$ problem we now see (by comparing with equation (5)) that we have

$$b_1^*=\frac{b_2-b_2'}{b_1-b_1'},\ b_2^*=\frac{b_3-b_3'}{b_1-b_1'},\ \dots,\ b_{m-2}^*=\frac{b_{m-1}-b_{m-1}'}{b_1-b_1'}$$
$$b_{m-1}^*=\frac{b_m}{b_1-b_1'} \quad \dots(12)$$

and

$$a_0^*=\frac{a_1-a_1'}{b_1-b_1'}=1,\ a_1^*=\frac{a_2-a_2'}{b_1-b_1'},\ a_2^*=\frac{a_3-a_3'}{b_1-b_1'},\ \dots,$$
$$a_{n-2}^*=\frac{a_{n-1}-a_{n-1}'}{b_1-b_1'},\ a_{n-1}^*=-\frac{a_n'}{b_1-b_1'}. \quad \dots(13)$$

A convenient form of the relations (13) are

$$a_0^*=1,\ a_1^*=1.\frac{a_2-a_2'}{a_1-a_1'},\ a_2^*=1.\frac{a_3-a_3'}{a_1-a_1'},\ \dots,$$
$$a_{n-2}^*=1.\frac{a_{n-1}-a_{n-1}'}{a_1-a_1'},\ a_{n-1}^*=-1.\frac{a_n'}{a_1-a_1'}. \quad \dots(14)$$

The recursion relations (12) and (14) enable us to construct the Padé table starting from its first row and first column. With this in view we write the equations for determining b's in the form

$$\left.\begin{aligned} b'_0 &= 1 \\ b'_i &= b_i - \frac{b^*_{i-1}\, b_m}{b^*_{m-1}},\ i = 1, 2, \ldots, m-1 \end{aligned}\right\} \qquad \ldots(15)$$

and a equations in the form

$$\left.\begin{aligned} a_0 &= 1 \\ a_i &= a'_i - \frac{a^*_{i-1}\, a'_n}{a^*_{n-1}} \end{aligned}\right\} \qquad \ldots(16)$$

The significance of the symbols $*$, $'$ and unprimed coefficients can be visualized by the scheme

$$\begin{matrix} * & ' \\ \square & \end{matrix} \qquad \ldots(17)$$

Here the symbol $\square$ denotes the unprimed coefficients and the three symbols are written above according to the relative positions in the Padé table. Equations (15) represent the solutions b' in terms of b and b^*. The scheme (17) indicates that from eqns. (15) we can calculate columns of b's in the Padé table successively from the first column of $(\overset{*}{\square})$. The a's are not solved in the same manner from eqns. (14). We can solve for a in terms of a' and a^* and the solution is given by eqns. (16). The scheme (17) shows that from eqns. (16) we calculate rows of a's in the Padé table successively from the first row of $(*\ ')$.

The first row of the Padé table consists simply of partial sums of $F(\alpha, \beta, \gamma; x)$ divided by unity

$$P_{0n} = 1 + \frac{\alpha.\beta}{1!\,\gamma}\, x + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)}\, x^2 + \ldots + \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\,\beta(\beta+1)\ldots(\beta+n-1)}{n!\,\gamma(\gamma+1)\ldots(\gamma+n-1)}\, x^n$$

$$Q_{0n} = 1. \qquad \ldots(18)$$

Hence if the first $(p + 1)$ coefficients $1, \frac{\alpha.\beta}{1!\,\gamma}, \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)},$

$$\ldots\ldots \frac{\alpha(\alpha+1)\ldots(\alpha+p-1)\,\beta(\beta+1)\ldots(\beta+p-1)}{p!\,\gamma(\gamma+1)\ldots(\gamma+p-1)} \text{ of } F(\alpha, \beta, \gamma; x)$$

are given we can compute the coefficients of the Padé table out to the triangle

$$\left.\begin{array}{llll} R_{00} & R_{01} & R_{02} \ldots R_{0p} \\ R_{10} & R_{11} & R_{12} \ldots R_{1,p-1} \\ R_{20} & R_{21} & R_{22} \ldots R_{2,p-2} \\ \ldots & \ldots & \ldots \ \ldots \ \ldots \\ R_{p0} & & \end{array}\right\} \quad \ldots(19)$$

if we can obtain the first column $R_{00}, R_{10}, \ldots, R_{p0}$. To determine this column we note that for $n = 0$ the equations (6) reduce to the triangular system

$$\begin{aligned} &1 = a_0 \\ &\frac{\alpha.\beta}{1.\gamma} b_0 + b_1 = 0 \\ &\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)} b_0 + \frac{\alpha.\beta}{1.\gamma} b_1 + b_2 = 0 \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\frac{\alpha(\alpha+1)\ldots(\alpha+m-1)\,\beta(\beta+1)\ldots(\beta+m-1)}{m!\,\gamma(\gamma+1)\ldots(\gamma+m-1)} b_0 \\ &\quad + \frac{\alpha(\alpha+1)\ldots(\alpha+m-2)\,\beta(\beta+1)\ldots(\beta+m-2)}{(m-1)!\,\gamma(\gamma+1)\ldots(\gamma+m-2)} b_1 \\ &\quad + \ldots + b_m = 0 \end{aligned} \quad \ldots(20)$$

and making use of (7) we can solve recursively by means of the scheme

$$\begin{aligned} b_0 &= 1 \\ b_1 &= -\frac{\alpha.\beta}{1!\,\gamma} \\ b_2 &= -\left[\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)} + \frac{\alpha.\beta}{1!\,\gamma} b_1\right] \\ &\vdots \\ b_p &= -\left[\frac{\alpha(\alpha+1)\ldots(\alpha+p-1)\,\beta(\beta+1)\ldots(\beta+p-1)}{p!\,\gamma(\gamma+1)\ldots(\gamma+p-1)}\right. \\ &\quad + \frac{\alpha(\alpha+1)\ldots(\alpha+p-2)\,\beta(\beta+1)\ldots(\beta+p-2)}{(p-1)!\,\gamma(\gamma+1)\ldots(\gamma+p-2)} b_1 \\ &\quad \left. + \ldots + \frac{\alpha.\beta}{1.\gamma} b_{p-1}\right]. \end{aligned} \quad \ldots(21)$$

Thus the first column of R_{m0} of the Padé table is given by

$$\left.\begin{aligned} P_{m0} &= 1 \\ Q_{m0} &= \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_m x^m \end{aligned}\right\} m = 0, 1, 2, \ldots, p \quad \ldots(22)$$

where the λ's are b's as given in eqn. (21). We have used a modified form of the Fortran Subroutine of Longman to compute the Padé table from the first $(n + 1)$ coefficients

$$1, \frac{\alpha.\beta}{1.\gamma}, \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)}, \ldots, \frac{\alpha(\alpha+1)\ldots(\alpha+n-2)\,\beta(\beta+1)\ldots(\beta+n-2)}{(n-1)!\,\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-2)}$$

of eqn. (1).

4. Applications

The elliptic integrals of the first and the second kind can be expressed in terms of hypergeometric functions through suitable choices of the parameters (α, β, γ). (Erdélyi 1953). For example, the elliptic integral of the first kind and second kind are given by

$$\int_0^{\pi/2} (1 - k^2 \sin^2\theta)\, d\theta = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1; k^2), \ |k| < 1 \qquad \ldots(23)$$

and

$$\int_0^1 \left(\frac{1 - k^2x^2}{1 - x^2}\right)^{1/2} dx = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1; k^2), \ |k| < 1. \qquad \ldots(24)$$

Table I

Coefficients of the Padé table	Values of the coefficients ($k = 0.5$)
R_{00}	1
R_{01}	1.0625000
R_{10}	1.0666666
R_{02}	1.0712890
R_{20}	1.0722513
R_{11}	1.0727272
R_{03}	1.0728149
R_{30}	1.0730237
R_{12}	1.0731355
R_{21}	1.0731470
R_{04}	1.0731070
R_{40}	1.0731525
R_{13}	1.0731761
R_{31}	1.0731782
R_{22}	1.0731795

We have evaluated the respective hypergeometric functions on the right-hand sides of eqns. (23) and (24) by their Padé approximants. Tables I and II give the values of the elements R_{mn} for $k = 0.5$ for the functions $F(\frac{1}{2}, \frac{1}{2}, 1; k^2)$ and $F(-\frac{1}{2}, \frac{1}{2}, 1; k^2)$ respectively. A good agreement is observed between the standard tabulated values of the elliptic function and the value R_{nn} multiplied by $\pi/2$.

The program was run on an IBM 1130 using extended precision arithmetic.

Value of $\int_0^{\pi/2} (1 - 0.25 \sin^2 \theta)^{-1/2} d\theta$ rounded to eight digits = 1.68575035 (Abramowitz and Stegun 1970). Value of $\frac{\pi}{2}.R_{22} = 1.68574641$.

TABLE II

Coefficients of the Padé table	Value of the coefficients ($k = 0.5$)
R_{00}	1
R_{01}	0.93750000
R_{10}	0.94117647
R_{02}	0.93457031
R_{20}	0.93515981
R_{11}	0.93442623
R_{03}	0.93426513
R_{30}	0.93435985
R_{12}	0.93422965
R_{21}	0.93423626
R_{04}	0.93422341
R_{40}	0.93423935
R_{13}	0.93421561
R_{31}	0.93421795
R_{22}	0.93421644

Value of $\int_0^{\pi/2} (1 - 0.25 \sin^2\theta)^{-1/2} d\theta$ rounded to eight digits = 1.46746221 (Abramowitz and Stegun 1970). Value of $\frac{\pi}{2}.R_{22} = 1.46746375$.

Tables III and IV give exclusively the diagonal elements of R_{nn} for $k = 0.5$ for the function $F(\frac{1}{2}, \frac{1}{2}, 1; k^2)$ and $F(-\frac{1}{2}, \frac{1}{2}, 1; k^2)$ respectively.

TABLE III

n	$R_{nn}(k = 0.5)$
0	1
1	1.07272720
2	1.07317955
3	1.07318199
4	1.07318200
5	1.07318200

TABLE IV

n	$R_{nn}(k = 0.5)$
0	1
1	0.93442622
2	0.93423626
3	0.93421546
4	0.93421545
5	0.93421545

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REFERENCES

Abramowitz, M., and Stegun, A. (1970). Handbook of Mathematical Functions. Ninth Printing, NBS, Appl. Math. Series 55, U.S.A.

Erdélyi, A. *et al.* (1953). Higher Transcendental Functions, Vol. I. McGraw-Hill Book Co., Inc., New York.

Fike, C. T. (1968). Computer Evaluation of Mathematical Functions. Prentice-Hall, Inc., New York.

Ince, E. L. (1956). Ordinary Differential Equations. Dover Publications, New York.

Longman, I. M. (1971). Computation of the Padé table. *Int. J. Computer Math.*, Sec. B, **3**, 53-64.

Luke, Y. L. (1969). The Special Functions and Their Approximations, Vol. 2. Academic Press, New York.

Luke, Y. L. (1970). Further Approximations for Elliptic Integrals. *Math. Comp.*, **24**, No. 109, 191-98.

Perron, O. (1950). Die Lehre von dem Kettenbrüchen. Chelsea, New York.