UNSTEADY FLOW THROUGH MAGNETOHYDRODYNAMIC POROUS MEDIA

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An exact solution of the unsteady motion of an electrically conducting, incompressible and viscous fluid through a porous medium under the action of a transverse magnetic field is obtained. The results are interpreted with the aid of graphs.

1. INTRODUCTION

Origin of the flow through porous media relies heavily upon Darcy's experimental law. The validity of this law is subject to several limitations. It is shown that a it can be possibly valid only in a certain "seepage" velocity domain outside which more general flow equations must be used to describe the flow correctly. This happens because the inertial effects become important. Recently, Ahmadi and Manvi (1971) derived a general equation of motion and applied the results obtained to some basic flow problems.

The aim of the present investigation is to study the unsteady motion of an electrically conducting, incompressible and viscous fluid through a porous medium in the presence of a transverse magnetic field. In particular, we have discussed the following problems :

(i) Unsteady MHD flow through a porous medium between two parallel plates.

(ii) Unsteady MHD flow through a porous medium in a circular pipe.

2. UNSTEADY MHD FLOW THROUGH A POROUS MEDIUM BETWEEN TWO PARALLEL PLATES

Formulation of the Problem

The porous material containing the fluid is in fact a non-homogeneous medium. For the sake of analysis, it is possible to replace it with a homogeneous fluid which has dynamical properties equal to the local averages of the original non-homogeneous continuum. Then, one can study the motion of the hypothetical homogeneous fluid under the action of the properly averaged external forces. Thus, the complicated problem of the motion of a viscous fluid in a porous solid reduces to the motion of the homogeneous fluid with some additional resistances.

We consider the unsteady motion of an electrically conducting, incompressible and viscous fluid through a porous medium in the presence of a transverse magnetic field. We select a rectangular Cartesian system with the axis of x in the direction of motion and axis of y perpendicular to it.

The governing equations are

$$\frac{\partial \rho}{\partial t} + \rho \nabla . \vec{V} = 0 \qquad ...(2.1)$$

$$\rho \frac{\partial V}{\partial t} = - \nabla p + \mu \nabla^2 \vec{V} - \mu \frac{\vec{V}}{k} + \vec{J} \times \vec{B} \qquad ...(2.2)$$

where ρ is the density; $\vec{V}$, the velocity vector; p, the pressure; μ, the fluid viscosity; K, the permeability of the medium; $\vec{J}$, the current density; and $\vec{B}$, the magnetic induction. As we assume that R_m, the magnetic Reynolds number is small, Maxwell's equations become redundant.

The above equations reduce to

$$\rho \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} - \mu \frac{u}{k} - \sigma B_0^2 u \qquad ...(2.3)$$

$$- \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \qquad ...(2.4)$$

$$\frac{\partial u}{\partial x} = 0 \qquad ...(2.5)$$

with the initial and the boundary conditions

$$\left.\begin{aligned} &u = 0; \quad \text{everywhere for } t \leqslant 0 \\ \text{and} \quad &u = 0; \quad \text{for } y = \pm h, \ t > 0. \end{aligned}\right\} \qquad ...(2.6)$$

Let us introduce the following non-dimensional quantities

$$\eta = \frac{y}{h},$$

$$t' = \frac{\nu t}{h^2},$$

$$v' = \frac{u}{U},$$

$$x' = \frac{x}{h},$$

$$p' = \left(\frac{ph}{\mu U}\right),$$

$$K' = \frac{k}{h^2}$$

and

$$M = hB_0 \sqrt{\frac{\sigma}{\mu}} \text{ (Hartmann number),} \qquad \ldots(2.7)$$

where U is the characteristic velocity.

By virtue of eqn. (2.7), eqns. (2.3), (2.4) and (2.5) become

$$\frac{\partial u'}{\partial t'} = -\frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial \eta^2} - \frac{u'}{k'} - M^2 u' \qquad \ldots(2.8)$$

$$\frac{\partial p'}{\partial \eta} = 0 \qquad \ldots(2.9)$$

$$\frac{\partial u'}{\partial x'} = 0 \qquad \ldots(2.10)$$

with the initial and the boundary conditions

$$\left.\begin{array}{ll} u' = 0; & \text{everywhere for } t' \leqslant 0 \\ \text{and} & \\ u' = 0, & \text{for } \eta = \pm 1, \quad t' > 0. \end{array}\right\} \qquad \ldots(2.11)$$

Solution of the Problem

From eqn. (2.9), it is clear that p' is independent of η and from eqn. (2.8) it follows that $(\partial p'/\partial x')$ is a function of t' only. Eqn. (2.10) shows that u' is independent of x', and therefore, a function of η and t' only.

Let

$$\frac{\partial p'}{\partial x'} = -f(t'). \qquad \ldots(2.12)$$

Eqn. (2.13) yields

$$\frac{\partial u'}{\partial t'} = f(t') + \frac{\partial^2 u'}{\partial \eta^2} - \frac{u'}{K'} - M^2 u'. \qquad \ldots(2.13)$$

Now, let us take $f(t') = B$ (a positive constant).

Then eqn. (2.13) becomes

$$\frac{\partial u'}{\partial t'} = B + \frac{\partial^2 u'}{\partial \eta^2} - \left(M^2 + \frac{1}{k'}\right) u'. \qquad \ldots(2.14)$$

On applying Laplace transform, eqn. (2.14) reduces to

$$\frac{d^2 \bar{u}'}{d\eta^2} - \left(M^2 + \frac{1}{k'} + S\right) \bar{u}' = -\frac{B}{S}. \qquad \ldots(2.15)$$

The conditions (2.11) transform to

$$\bar{u}' = 0; \quad \text{for } \eta = \pm 1$$

where

$$\bar{u}'(\eta, t') = \int_0^\infty u'(\eta, t')\, e^{-St'} dt' \text{ and } S > 0. \qquad \ldots(2.16)$$

Solution of eqn. (2.15) with the aid of the conditions (2.16) yields

$$\bar{u}' = \frac{B}{S\left(S + M^2 + \frac{1}{k'}\right)} \left[1 - \frac{\cosh(\sqrt{S + M^2 + k})\eta}{\cosh\sqrt{S + M^2 + \frac{1}{k'}}}\right]. \qquad \ldots(2.17)$$

On applying inversion formula, eqn. (2.17) yields

$$u'(\eta, t') = \frac{B}{\left(M^2 + \frac{1}{K}\right)} \left[1 - \frac{\cosh\sqrt{\left(M^2 + \frac{1}{k'}\right)}\,\eta}{\cosh\sqrt{M^2 + \frac{1}{K'}}}\right]$$

$$- \frac{16B}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r e^{-\left\{\frac{\pi^2(2r+1)^2 + 4(M^2 + (1/k'))}{4}\right\} t'}}{(2r+1)\left\{(2r+1)^2 + 4\left(M^2 + \frac{1}{K'}\right)\right\}}$$

$$\times \cos\left\{\frac{\pi(2r+1)}{2}\,\eta\right\}. \qquad \ldots(2.18)$$

For $t' \to \infty$ and $M \to 0$, the solution (2.18) reduces to the case of Ahmadi and Manvi (1971)

$$u' = \frac{B}{\frac{1}{K}} \left[1 - \frac{\cosh\sqrt{\frac{1}{k'}}\,\eta}{\cosh\sqrt{\frac{1}{K'}}}\right]. \qquad \ldots(2.19)$$

Again, (2.19) reduces to the usual laminar flow between two parallel flat plates for $K' \to \infty$,

$$u' = \tfrac{1}{2}B(1 - \eta^2) \qquad \text{(Hagen-Poiseuille flow).} \qquad \ldots(2.20)$$

Discussion

In Fig. 1, the velocity profiles have been drawn for different values of M and K' by taking $t' = 0.5$. It is clear from Fig. 1 that increase in K' accelerates the flow. Fig. 2 shows difference between the steady and the unsteady motions. It is obvious that the unsteady motion converts into steady flow after considerable lapse of time. It is evident from Fig. 3 that the flow becomes more stable near the walls.

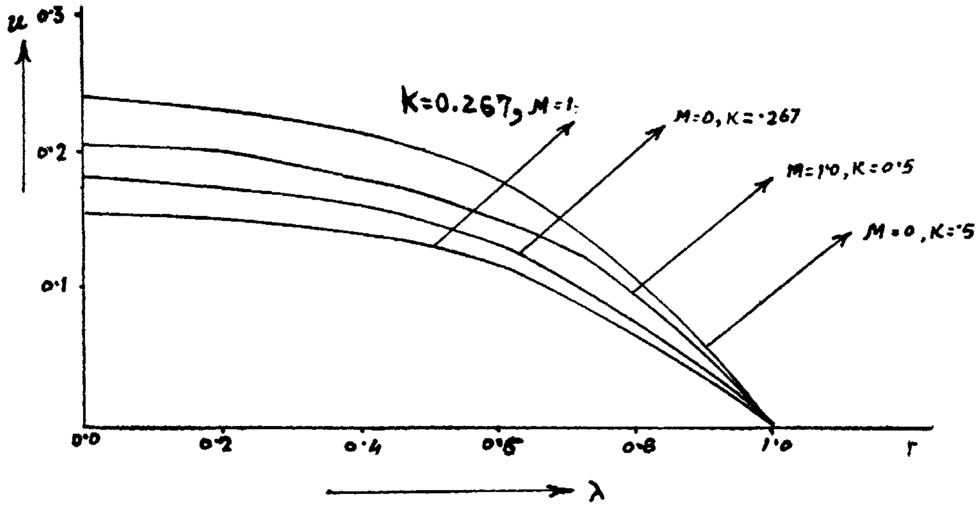

FIG. 1. Velocity profile for $t = 0.5$, $P = 1.0$.

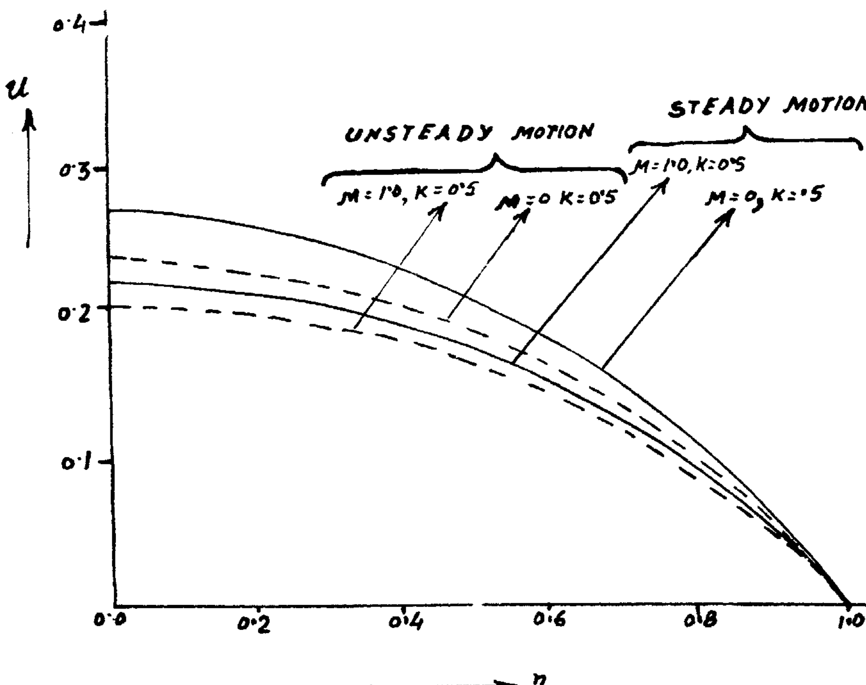

FIG. 2. Velocity profile for $t = 0.5$.

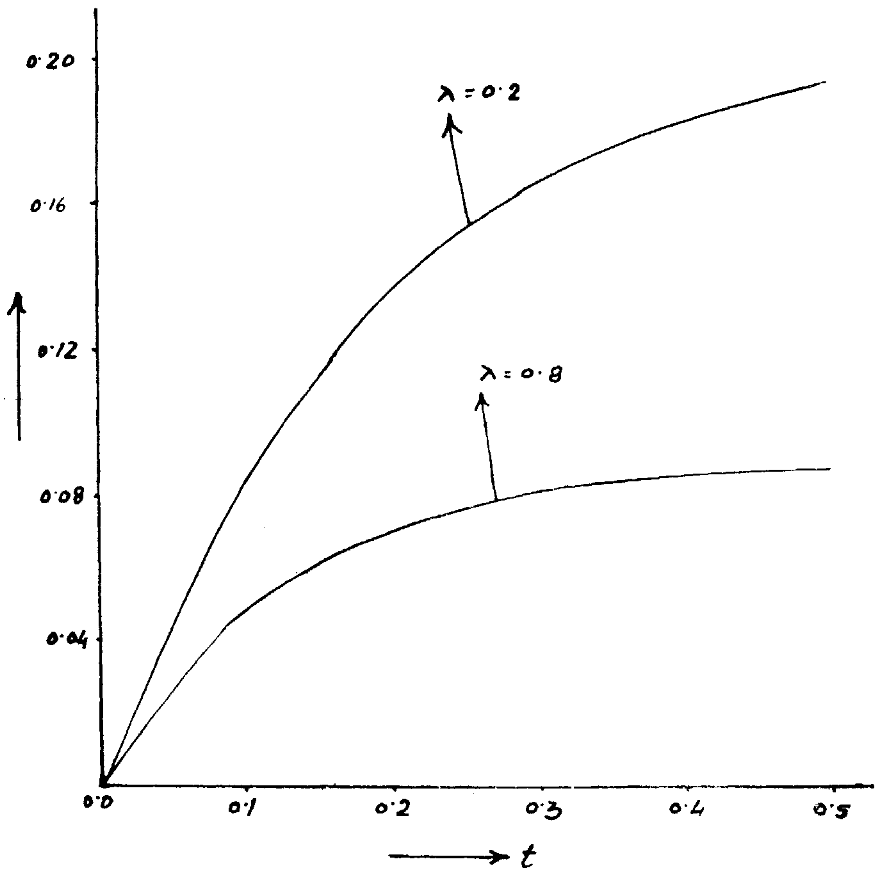

FIG. 3. Velocity profile for $P = 1.0$, $k = 0.5$, $M = 0$.

3. Unsteady MHD Flow Through a Porous Medium in a Circular Pipe

Formulation of the Problem

Consider the unsteady motion of an electrically conducting, incompressible and viscous fluid through a porous medium in a circular pipe in the presence of a magnetic field acting along the radius of the pipe. We select a cylindrical polar coordinate system with z-axis in the direction of motion.

The governing equations are

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} - \mu\frac{u}{K} + \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) - \sigma B_0^2 u \qquad \dots(3.1)$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \qquad \dots(3.2)$$

$$\frac{\partial u}{\partial z} = 0 \qquad \dots(3.3)$$

with the initial and the boundary conditions

$$\begin{aligned} & u = 0, \quad \text{everywhere for } t \leqslant 0, \\ & \left. \begin{aligned} & u = 0, \quad r = a \\ & u = \text{finite}, r = 0 \end{aligned} \right\} \text{for } t > 0. \end{aligned} \qquad \ldots(3.4)$$

Let us introduce the following non-dimensional quantities

$$\left. \begin{aligned} \lambda &= \frac{r}{a}, \\ t' &= \frac{\nu t}{a^2}, \\ u' &= \frac{u}{U}, \\ z' &= \frac{z}{a}, \\ p' &= \left(\frac{p}{\frac{\mu U}{a}} \right), \\ k' &= \frac{k}{a^2}, \\ \text{and} \\ M &= aB_0 \sqrt{\frac{\sigma}{\mu}} \text{ (Hartmann number)}, \end{aligned} \right\} \qquad \ldots(3.5)$$

where U is the characteristic velocity.

In view of these, eqns. (3.1), (3.2) and (3.3) become

$$\frac{\partial u'}{\partial t'} = -\frac{\partial p'}{\partial z'} + \frac{\partial^2 u'}{\partial \lambda^2} + \frac{1}{\lambda} \cdot \frac{\partial u'}{\partial \lambda} - \left(M^2 + \frac{1}{k'} \right) u' \qquad \ldots(3.6)$$

$$\frac{\partial p'}{\partial \lambda} = 0 \qquad \ldots(3.7)$$

$$\frac{\partial u'}{\partial z'} = 0 \qquad \ldots(3.8)$$

with the initial and the boundary conditions

$$\left. \begin{aligned} & u' = 0, \quad \text{everywhere} \quad \text{for } t' \leqslant 0, \\ & \left. \begin{aligned} & u' = 0, \quad \lambda = 1 \\ & u' = \text{finite}, \lambda = 0 \end{aligned} \right\} \quad \text{for } t' > 0. \end{aligned} \right\} \qquad \ldots(3.9)$$

Equation (3.8) shows that u' is independent of z', and, therefore, a function of λ and t' only. From eqn. (3.7), it is clear that p' is independent of λ and from eqn. (3.6) it follows that $(\partial p'/\partial z')$ is a function of t' only.

Solution of the Problem

Let

$$\frac{\partial p'}{\partial z'} = - f(t') = C \qquad \text{(a positive constant).} \qquad \ldots(3.10)$$

Then eqn. (1.6) yields

$$\frac{\partial u'}{\partial t'} = C + \frac{\partial^2 u'}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial u'}{\partial \lambda} - \left(M^2 + \frac{1}{k'}\right) u' \qquad \ldots(3.11)$$

On applying Laplace transform, eqn. (3.11) reducecs to

$$\frac{d^2 \bar{u}'}{d\lambda^2} + \frac{1}{\lambda}\frac{d\bar{u}'}{d\lambda} - \left(M^2 + \frac{1}{K} + S\right) + \bar{u}' = - \frac{C}{S}, \qquad \ldots(3.12)$$

where

$$\bar{u}' = \int_0^\infty u' e^{-St'} \, dt'$$

and $S > 0$. ...(3.13)

The boundary conditions (3.9) transform to

$$\bar{u}' = 0, \qquad \lambda = 1$$

$$\bar{u}' = \text{finite}; \ \lambda = 0. \qquad \ldots(3.14)$$

Solving eqn. (3.12) with the aid of (3.14), we get

$$\bar{u}' = \frac{C}{S\left(S + M^2 + \frac{1}{k'}\right)} \left[1 - \frac{I_0\left(\left\{\sqrt{M^2 + \frac{1}{k'} + S}\right\}\lambda\right)}{I_0\left(\cdot\sqrt{M^2 + \frac{1}{k'} + S}\right)}\right]. \qquad \ldots(3.15)$$

Applying Laplace inversion, eqn. (3.15) yields

$$u' = \frac{C}{\left(M^2 + \frac{1}{k'}\right)} \left[1 - \frac{I_0\left\{\sqrt{\left(M^2 + \frac{1}{k'}\right)}\lambda\right\}}{I_0\left\{\sqrt{M^2 + \frac{1}{k'}}\right\}}\right]$$

$$- 2C \sum_i \frac{J_0(\xi_i \lambda) \, e^{-\left\{\xi_i^2 + M^2 + \frac{1}{k'}\right\} t'}}{\xi_i \left(\xi_i^2 + M^2 + \frac{1}{k'}\right) J_0(\xi_i)} \qquad \ldots(3.16)$$

where ξ_i are the roots of $J_0(\xi_i)$.

The solution (3.16) reduces to the case of Ahmedi and Manvi (1971) for $t' \to \infty$ and $M \to 0$,

$$u' = \frac{C}{\frac{1}{k'}} \left[1 - \frac{I_0 \left\{ \sqrt{\frac{1}{k'}} \cdot \lambda \right\}}{I_0 \left\{ \sqrt{\frac{1}{k'}} \right\}} \right]. \quad \ldots(3.17)$$

Again, (3.17) reduces to the steady laminar flow in a circular pipe $k' \to \infty$,

$$u' = \frac{C}{4}(1 - \lambda^2) \quad \text{(Hagen-Poiseuille flow)} \quad \ldots(3.18)$$

Discussion

From Fig. 4 it is clear that in hydrodynamic case velocity gradually increases with increase in time and it approximately reaches the steady motion for every $t' \geqslant 0.5$. A similar conclusion can be drawn in the case of magnetohydrodynamic flow (Fig. 5). But the flow in this case is more rigid.

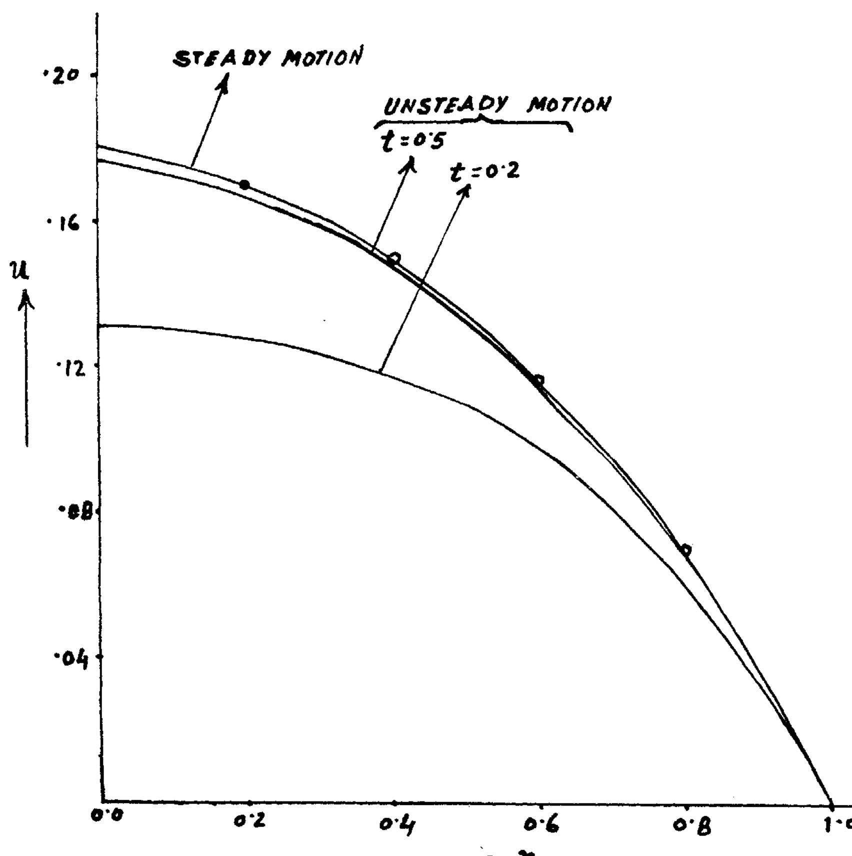

FIG. 4. Velocity profile for $M = 0$, $k = 0.5$, $C = 1.0$.

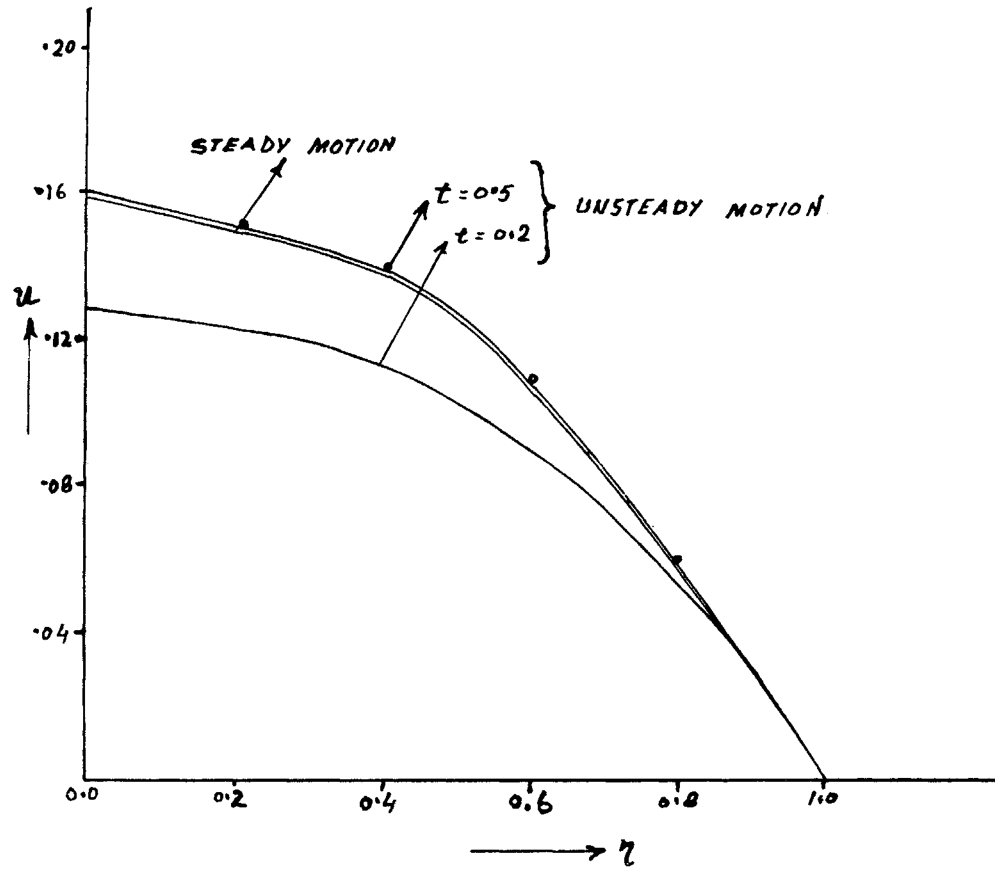

FIG. 5. Velocity profile for $M = 1.0$, $k = 0.5$, $C = 1.0$.

4. APPLICATIONS

The study of physics of flow through porous media has become the basis for many scientific and engineering applications. This type of flow is of great importance to the petroleum engineer concerned with the movement of oil, gas and water through the reservoir of an oil or gas field, to the hydrologist in his study of the migration of underground water, and to the chemical engineer in connection with filtration processes. Beyond this, the study is widely applicable in soil mechanics, water purification, ceramic engineering and powder metallurgy.

The results of the problem are also of great interest in geophysics in the study of the interaction of the geomagnetic field with the fluid in the geothermal region. Water in the geothermal region is an electrically conducting liquid because of high temperature. With the fuel crisis deepening all over the developed world, attention is turning to the utilization of the enormous power beneath the earth's crust in the geothermal region.

Another potential geophysical application of the present results is in the exploration of geopressured reservoirs. In these reservoirs, water at elevated temperature exists at enormously high pressure because of the weight of the overlying rock and the geomagnetic field. The upflowing water from geopressured wells can run hydraulic turbines to produce electricity, while the heat in the water can simultaneously be exracted to run steam turbines, again producing electricity.

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