ELECTROHYDRODYNAMIC STABILITY OF A VISCOUS CIRCULAR INTERFACE

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The electrohydrodynamic stability of two dielectric viscous fluids confined in an infinite rigid cylinder and having a circular interface has been investigated. It is found that there exists a critical electric field beyond which unstable regions appear. The eigen wavenumbers which bound the unstable regions have been determined theoretically and numerically. It is also found that some stable wave numbers for the infinite interface are not allowed in the case of circular interface. The value of the electric field in which unstable deformations are set up at lower eigenwave numbers has been calculated.

1. INTRODUCTION

The literature in the field of electrohydrodynamics is concerned mainly with surface phenomena such as charge relaxations (Chandrasekhar 1961), and surface waves (Malkus and Veronis 1961). Some interest has also been shown in bulk waves and thin films (Melcher 1963, Nayyar and Murty 1960) and electrohydrodynamic flow past obstacles.

In this paper, we have considered the influence of an initially perpendicular electric field on a circular interface between two superposed viscous fluids confined in an infinite rigid cylinder. We have derived the dispersion relation and have related the general case to less involved limiting situations. Eigenvalues of wavenumbers due to the presence of the rigid cylinder have been determined both theoretically and numerically.

It has been found that the electric field has a destabilizing effect and there exists a critical normalized electric pressure $\bar{P} = \sqrt{2}$, such that for $\bar{P} > \sqrt{2}$, unstable modes are observed for the set of wavenumbers having values between k^{cl}_{mn} and k^{cu}_{mn}.

2. THE EQUATIONS OF MOTION

The system under study is a small amplitude motion of a circular interface between two incompressible viscous fluids. The two dielectric fluids are confined

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in an infinite cylinder ($r = R$) and separated by the plane $z = 0$ (using cylindrical coordinates r, θ, z). On either side of the interface, the fluids are homogeneous with conductivities σ^u and σ^l, densities ρ^u and ρ^l, viscosities μ^u and μ^l and uniform vertical fields E_0^u and E_0^l. The superscripts u and l refer to the upper and the lower region respectively. The gravitational acceleration g acts in the negative z direction. The equations governing motion are

$$\nabla \cdot \epsilon \mathbf{E} = 0 \qquad \ldots(1)$$

$$\nabla \wedge \mathbf{E} = 0 \qquad \ldots(2)$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla \pi + \nabla \cdot \overleftrightarrow{T} - \rho \mathbf{g} \qquad \ldots(3)$$

$$\nabla \mathbf{v} \cdot = 0 \qquad \ldots(4)$$

$$\nabla \cdot \mathbf{J}_f + \frac{\partial q}{\partial t} = 0 \quad \text{or} \quad \nabla \cdot (\sigma \mathbf{E} + q\mathbf{v}) + \frac{\partial q}{\partial t} = 0 \qquad \ldots(5)$$

where $\mathbf{J}_f$ is the free current density, ϵ the permitivity, and $\overleftrightarrow{T}$ the stress tensor expressed in tensor notations as

$$T_{ij} = \epsilon E_i E_j - \tfrac{1}{2}\delta_{ij} \epsilon E_k E_k + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad \ldots(6)$$

$$\pi = p - \tfrac{1}{2}\rho \left.\frac{\partial \epsilon}{\partial \rho}\right|_T E^2.$$

The equilibrium solution of eqn. (3) is,

$$\pi_0(z) = -\rho g z + \Pi. \qquad (\Pi \text{ is a constant}) \qquad \ldots(7)$$

We now imagine that the circular interface is perturbed and the surface of the deformed interface is given by $z = \xi$, where

$$\xi = \delta f(r) \exp\left[i(m\theta + \omega t)\right] \qquad \ldots(8)$$

and δ is a smallness parameter having the dimensions of length. Consequently, the dependent variables assume the form

$$\pi = \pi_0 + \pi_1, \ \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1, \ \mathbf{v} = \mathbf{v}_1.$$

The motion can then be described by the following linearized equations :

$$\rho \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla \pi_1 + \mu \nabla^2 \mathbf{v}_1 \qquad \ldots(9)$$

$$\nabla \cdot \mathbf{v}_1 = 0 \qquad \ldots(10)$$

$$\nabla \cdot \epsilon \mathbf{E}_1 = 0 \qquad \ldots(11)$$

$$\triangle \wedge \mathbf{E}_1 = 0 \qquad \ldots(12)$$

for both regions of the fluids.

Since $\mathbf{E}_1$ is irrotational, there exists a scalar ϕ_1, such that $\mathbf{E}_1 = - \nabla \phi_1$ and from eqn. (11), we find that ϕ_1 satisfies

$$\nabla^2 \phi_1 = 0. \qquad ...(13)$$

Also from eqns. (9) and (10), we get

$$\nabla^2 \pi_1 = 0. \qquad ...(14)$$

Assuming the space and time dependence of the perturbed quantities to be of the type

$$F(r, \theta, z) = f(r, z) \exp [i(m\theta + \omega t)]$$

we can solve eqns. (13) and (14) to get the following solutions

$$\phi_1^{u,l} = C^{u,l} J_m(kr) \exp [i(m\theta + \omega t) \mp kz] \qquad ...(15)$$

$$\pi_1^{u,l} = A^{u,l} J_m(kr) \exp [i(m\theta + \omega t) \mp kz] \qquad ...(16)$$

where $J_m(kr)$ is the Bessel function of order m of the first kind. The arbitrary constants $C^{u,l}$ and $A^{u,l}$ are of order δ.

We may use the transformation

$$i\omega\rho \mathbf{v}_1 + \nabla \pi_1 = i\omega\rho \mathbf{v}_{11}.$$

The solution for eqn. (9) is

$$v_{1r}^{u,l} = \frac{iA^{u,l}k}{\omega\rho^{u,l}} J_m(kr)\, e^{\mp kz} + \frac{\mp \beta^{u,l} B^{u,l}}{k} J'_m(kr)\, e^{\mp \beta^{u,l} z} + \frac{F^{u,l}}{r\beta^{u,l}} J_m(kr)\, e^{\mp \beta^{u,l} z} \qquad ...(17)$$

$$v_{1\theta}^{u,l} = - \frac{A^{u,l}m}{\omega r\rho^{u,l}} J_m(kr)\, e^{\mp kz} + \frac{\mp \beta^{u,l} imB^{u,l}}{k^2 r} J_m(kr)\, e^{\mp \beta^{u,l} z} + \frac{iF^{u,l}k}{m\beta^{u,l}} J'_m(kr)\, e^{\mp \beta^{u,l} z} \qquad ...(18)$$

$$v_{1z}^{u,l} = \frac{\mp ikA^{u,l}}{\omega\rho^{u,l}} J_m(kr)\, e^{\mp kz} + B^{u,l} J_m(kr)\, e^{\mp \beta^{u,l} z} \qquad ...(19)$$

where $\beta^{u,l} = k^2 + R_1^{u,l}$, $R_1^{u,l} = i\rho^{u,l}/\mu^{u,l}$ and the prime denotes differentiation with respect to the argument.

3. Boundary Conditions

There are certain boundary conditions which should be satisfied at the boundaries $r = R$ and $z = \xi$.

The radial velocity v_{1r} should vanish at $r = R$. Therefore $F^{u,l} = 0$

and

$$J'_m (kR) = 0, \qquad \ldots(20)$$

i.e., k should assume a discrete set of eigenvalues k_{mn} determined by the roots of eqn. (20), which can be given by McMahon's formula (1945).

$$k_{mn} = \beta' - \frac{m' + 3}{8\beta'} - 4\,\frac{7m'^2 + 82m' - 9}{3(8\beta')^3} - \frac{32(83m'^3 + 2075m'^2 + 3039m' + 3527)}{15(8\beta')^5} + \ldots \qquad \ldots(21)$$

where $m' = 4m^2$, $\beta' = \frac{1}{4}\pi(2m + 4n + 1)$, $R = 1$.

The normal velocity at the interface $\partial\xi/\partial t$ should be continuous at the interface $z = \xi$ and, therefore, it should be uniquely expressed by the solutions for v_{1r}, whence

$$-\frac{ik_{mn}A^u}{\omega\rho^u} - i\,\frac{k_{mn}A^l}{\omega\rho^l} + B^u - B^l = 0. \qquad \ldots(22)$$

The continuity of the tangential velocity at the interface $z = \xi$ implies that

$$\frac{ik_{mn}A^u}{\omega\rho^u} - \frac{ik_{mn}A^l}{\omega\rho^l} - \frac{\beta^u}{k_{mn}}B^u - \frac{\beta^l}{k_{mn}}B^l = 0. \qquad \ldots(23)$$

The tangential component of the electric field is continuous at the interface $z = \xi$; therefore,

$$\frac{A^u k_{mn}(E_0^u - E_0^l)}{\omega^2\rho^u} - \frac{B^u(E_0^u - E_0^l)}{i\omega} + C^u - C^l = 0. \qquad \ldots(24)$$

The condition for charge conservation (Melcher and Smith 1969), namely,

$$\vec{N}\cdot(\sigma^u\mathbf{E}^u - \sigma^l\mathbf{E}^l) + \nabla_\Sigma\cdot Q\mathbf{v} + \frac{\partial Q}{\partial t} = 0$$

where $\vec{N}$ is the unit normal to the interface $z = \xi$, ∇_Σ is the surface derivative and Q is the surface charge density, leads to the following relation

$$-\frac{ik_{mn}^2}{\omega\epsilon^u}(\epsilon^u E_0^u - \epsilon^l E_0^l)\,A^u + \beta^u(\epsilon^u E_0^u - \epsilon^l E_0^l)\,B^u + k_{mn}(\sigma^u + i\omega\epsilon^u)\,C^u + k_{mn}(\sigma^l + i\omega\epsilon^l)\,C^l = 0. \qquad \ldots(25)$$

The continuity of the tangential stress at the interface $z = \xi$ yields

$$A^u\left[\frac{k_{mn}^2}{\omega^2\rho^u}(\epsilon^u {E_0^u}^2 - \epsilon^l {E_0^l}^2) + \frac{2k_{mn}^2\,\mu^u i}{\omega\rho^u}\right] + A^l\,\frac{2k_{mn}^2\,\mu^l i}{\omega\rho^l}$$
$$+ B^u\left[-\frac{k_{mn}}{i\omega}(\epsilon^u {E_0^u}^2 - \epsilon^l {E_0^l}^2) - \frac{\mu^u {\beta^u}^2}{k_{mn}} - \mu^u k_{mn}\right]$$
$$+ B^l\left(\frac{\mu^l {\beta^l}^2}{k_{mn}} + \mu^l k_{mn}\right) + k_{mn}\epsilon^u E_0^u\, C^u - k_{mn}\epsilon^l E_0^l\, C^l = 0. \qquad \ldots(26)$$

Finally, the balance of the normal stress at the interface $z = \xi$ gives

$$\frac{ik_{mn}}{\omega\rho^u}A^u\left(-\frac{i\omega\rho^u}{k_{mn}} + \frac{f}{i\omega} - 2\mu^u k_{mn}\right) + \frac{ik_{mn}}{\omega\rho^l}A^l\left(\frac{i\omega\rho^l}{k_{mn}} + 2\mu^l k_{mn}\right)$$
$$+ B^u\left(-\frac{f}{i\omega} + 2\mu^u\beta^u\right) + 2B^l\mu^l\beta^l$$
$$+ k_{mn}C^u E_0^u\,\epsilon^u + k_{mn}C^l E_0^l\,\epsilon^l = 0 \qquad \ldots(27)$$

where $f = g(\rho^u - \rho^l) - Tk_{mn}^2$ and T is the surface tension.

The compatibility condition requires that the determinant of the coefficient $A^{u,l}$, $B^{u,l}$, $C^{u,l}$ should vanish. The resulting seventh degree equation is

$$-\left[1 + gk_{mn}\frac{\alpha^l - \alpha^u}{S^2} + \frac{Tk_{mn}^3}{S^2(\rho^u + \rho^l)} + k_{mn}^2\,\frac{\phi(E_0^u - E_0^l)}{S^2(\rho^u + \rho^l)}\right.$$
$$\left. + \frac{e_{mn}k_{mn}}{d_{mn}}\right]\cdot\left[\frac{Se_{mn}}{k_{mn}} + \frac{\theta_1 d_{mn}(\epsilon^u E_0^u - \epsilon^l E_0^l)}{\rho^u + \rho^l}\right]$$
$$+\left[k_{mn}\psi\,\frac{\epsilon^u E_0^u - \epsilon^l E_0^l}{S^2(\rho^u + \rho^l)} + \frac{2k_{mn}(\mu^u - \mu^l)}{S(\rho^u + \rho^l)} - \frac{h_{mn}}{d_{mn}}\right]$$
$$\times\left[-h_{mn}S + \frac{2d_{mn}k_{mn}(\mu^u - \mu^l)}{\rho^u + \rho^l} + \theta_1 k_{mn}d_{mn}\,\frac{\epsilon^u E_0^u + \epsilon^l E_0^l}{\rho^u + \rho^l}\right] = 0 \qquad \ldots(28)$$

where

$$\alpha^u = \frac{\rho^u}{\rho^u + \rho^l},\ \alpha^l = \frac{\rho^l}{\rho^u + \rho^l},\ S = i\omega,$$
$$e_{mn} = \alpha^u(\beta^l - k_{mn}) + \alpha^l(\beta^u - k_{mn})$$
$$h_{mn} = \alpha^u(\beta^l - k_{mn}) + \alpha^l(\beta^u - k_{mn}),\ d_{mn} = (\beta^u - k_{mn})(\beta^l - k_{mn})$$

$$\phi = [\epsilon^l E_0^l (S\epsilon^u + \sigma^u) - \epsilon^u E_0^u (S\epsilon^l + \sigma^l)]/[S(\epsilon^u + \epsilon^l) + \sigma^u + \sigma^l]$$

$$\theta_1 = (\epsilon^u E_0^u - \epsilon^l E_0^l)/[S(\epsilon^u + \epsilon^l) + \sigma^u + \sigma^l]$$

$$\psi = [E_0^u (S\epsilon^u + \sigma^u) + E_0^l (S\epsilon^l + \sigma^l)]/[S(\epsilon^u + \epsilon^l) + \sigma^u + \sigma^l].$$

Eqn. (28) is similar to that obtained by Melcher and Smith (1969) for the case of an infinite interface, except that in the latter case, k and consequently e, d assume a continuous spectrum of values.

4. Stability Analysis

In the absence of electric field, the dispersion equation (28) reduces to that discussed by Chandrasekhar (1961), except that in the latter relation, k assumes a continuous spectrum of values. The applicability of eqn. (28) for irristricted k is discussed elsewhere (Melcher and Smith 1969). Therefore, we are concerned here with the effect of the circular boundary.

We note that in the limit $\mu^{u,l} \to 0$, eqn. (28) reduces to that obtained by Mohamed and Nayyar (1973) in their relation for the inviscid case.

If we take $\beta^u_{mn} \approx \beta^l_{mn}$, while $\mu^u \ll \mu^l$, then eqn. (28) becomes

$$\bar{\beta}^4_{mn} \bar{M}^2 + \bar{\beta}^2_{mn} (2\bar{k}^2_{mn} \bar{M}^2) + \bar{\beta}_{mn}(-4\bar{k}^3_{mn} \bar{M}^2) + \bar{k}^4_{mn} \bar{M}^2 + (\bar{k}_{mn} + \bar{k}^3_{mn} - \bar{k}^2_{mn} \bar{P}^2) = 0 \qquad \ldots(29)$$

where

$$\bar{\beta}_{mn} = \frac{\beta_{mn}}{k^*}, \quad \bar{S} = i\omega\left(\frac{T}{\rho g^3}\right)^{1/2}, \quad \bar{M} = \mu\left(\frac{g}{\rho T^3}\right)^{1/4}, \quad \bar{k} = \frac{k}{k^*}$$

$$\bar{P} = V_1\left(\frac{\rho}{k^* T}\right)^{1/2}, \quad V_1^2 = k^*\left(\frac{2T}{\rho^u + \rho^l}\right), \quad k^* = \left(\frac{g(\rho^u - \rho^l)}{T}\right)^{1/2}.$$

We solve eqn. (29) numerically for $\rho^l = \rho = 0.87 \times 10^3$ kg/m^3, $T = 4 \times 10^{-2}$ N/m, $g = 9.80665$ m/sec^2, $k^* = 461.95$ ($\rho^u = 0$) for various values of the normalized electric pressure $\bar{P}$ and viscosity $\bar{M}$ for restricted wavenumbers satisfying eqn. (20). We find that viscosity does not change the transition wavenumber. When $\bar{M}$ is held fixed, while $\bar{P}$ is changing, we observe that for $\bar{P} < \sqrt{2}$, the roots of eqn. (29) for $\bar{S}$ are either negative or have a negative real part and hence the system is stable. For $\bar{P} > \sqrt{2}$, we observe that unstable regions appear for such values of $\bar{k}_{mn}$ that

$\bar{k}^{cl}_{mn} \leqslant \bar{k}_{mn} \leqslant \bar{k}^{cu}_{mn}$. For example, if $\bar{P} = 2$, the system is stable for eigenwavenumbers less than $\bar{k}^{cl}_{mn} = \bar{k}^{cl}_{0\,39} = 0.273722$ ($\bar{k}^{cl}_{0\,39} = 126.446$) for symmetric deformations ($m = 0$). As $\bar{k}_{0\,n}$ becomes more than $k^{cl}_{0\,n}$ the system becomes unstable until we reach the last unstable wavenumber $\bar{k}^{cu}_{0\,n} = \bar{k}^{cu}_{0\,547} = 3.72849$ ($k^{cu}_{0\,547} = 1722.37$). For eigenvalues $\bar{k}_{0\,n}$ exceeding $\bar{k}^{cu}_{0\,547}$, the system is again stable. Increase in the value of $\bar{P}$ leads to increase in the unstable interval. The previous results can be justified, since the viscosity does not change the transition wavenumber. We see from eqn. (29) that in the limit of vanishing viscosity

$$\bar{k}_{mn} + \bar{k}^3_{mn} - \bar{k}^2_{mn}\,\bar{P}^2 = 0. \qquad \ldots(30)$$

Since $\bar{S} \to 0$ as $\bar{M} \to 0$ ($\bar{S} = \bar{M}(\beta^2_{mn} - \bar{k}^2_{mn})$), the roots of eqn. (30) determine $\bar{k}^{cl}_{mn}$ and $\bar{k}^{cu}_{mn}$. This result is in agreement with the above calculations. Moreover, the roots of eqn. (30) are complex conjugates for $\bar{P} < \sqrt{2}$. Since $\bar{k}_{mn}$ is real, there are no unstable regions for $\bar{P} < \sqrt{2}$. This explains why unstable regions appear for $\bar{P} > \sqrt{2}$ in our computation. The increase in $\bar{P}$ changes the values of the roots of eqn. (30) and the unstable intervals increase (since $\bar{k}^{cu}_{mn} - \bar{k}^{cl}_{mn} \simeq (\bar{P}^2 - 4)^{1/2}$). For $\bar{P} = 10$, the roots of eqn. (30) are 0.009 and 99.987, while in our computations for eqn. (29), we find that $\bar{k}^{cl}_{0n} = \bar{k}^{cl}_{01} = 0.0151868$. The reason is that 0.009 is not an element of the set of the eigenvalues given by eqn. (20), while 0.151868 is the smallest element of the set which exceeds 0.009.

We note that the first set of stable wavenumbers appearing in the case of infinite interface may not be allowed for the circular interface, if these numbers are smaller than the smallest eigenwavenumber determined from eqn. (20), namely $0.00398528 = \bar{k}_{1\,0}$. This emphasizes the destabilising nature of the circular boundary. The corresponding electric pressure for $\bar{k}^{cl}_{1\,0} = 0.00398528$ is $\bar{P} = 15.8$. Thus, for $\bar{P} > 15.8$, the first allowed eigenvalue is unstable and the band of unstable spectrum of eigenwavenumbers starts first.

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