INVARIANTS OF CURVATURE TENSOR AND GRAVITATIONAL RADIATION IN GENERAL RELATIVITY*

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Philip (1972) has worked out the invariants which characterize the curvature tensor in general relativity. On the other hand, Sharma and Husain (1969) and several other authors have given algebraic classifications of the curvature tensor in general relativity and have specified the cases which correspond the gravitational radiation. In the present paper, we have studied the forms of these invariants in the various cases of the classification of Sharma and Husain (1969) and that of Petrov, and have concluded that the gravitational radiation is characterized by the vanishing of all these invariants without curvature tensor being zero. This assertion is then verified by several known solutions representing gravitational radiation. In the last section, the 'super energy' of Bel (1960) is expressed in terms of some invariants and it is shown that the gravitational radiation is characterized by the vanishing of the super energy without curvature tensor being zero.

§1. (a) The curvature tensor has been classified by Sharma and Husain (1969) according to the number of eigen values of a complex six-dimensional tensor defined in terms of curvature tensor. The various cases that have been arrived at in that paper are summarized as under:

Case I — When all the three eigen values are different, the matrix for R_{abcd} reduces to

$$R_{abcd} = \begin{bmatrix} a_1 & 0 & 0 & b_1 & 0 & 0 \\ 0 & a_2 & 0 & 0 & b_2 & 0 \\ 0 & 0 & a_3 & 0 & 0 & b_3 \\ b_1 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & b_2 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & b_3 & 0 & 0 & -a_3 \end{bmatrix}$$

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Case II — When two of the three eigen values are equal, we have the following two cases :

Case II (*a*)

$$R_{abcd} = \begin{bmatrix} a_1 & 0 & 0 & b_1 & 0 & 0 \\ 0 & a_1 & 0 & 0 & b_1 & 0 \\ 0 & 0 & -2a_1 & 0 & 0 & -2b_1 \\ b_1 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & b_1 & 0 & 0 & -a_1 & 0 \\ 0 & 0 & -2b_1 & 0 & 0 & 2a_1 \end{bmatrix}$$

Case II (*b*)

$$R_{abcd} = \begin{bmatrix} 2a & 0 & 0 & 2b & 0 & 0 \\ 0 & -(a+d) & -c & 0 & (b-c) & -d \\ 0 & -c & -(a-d) & 0 & -d & (b+c) \\ 2b & 0 & 0 & -2a & 0 & 0 \\ 0 & (b-c) & -d & 0 & (a+d) & c \\ 0 & -d & (b+c) & 0 & c & (a-d) \end{bmatrix}$$

Case III — When all the three eigen values are equal, we have the following two possibilities :

Case III (*a*)

$$R_{abcd} = \begin{bmatrix} 0 & -a & -b & 0 & -b & a \\ -a & 0 & 0 & -b & 0 & 0 \\ -b & 0 & 0 & a & 0 & 0 \\ 0 & -b & a & 0 & a & b \\ -b & 0 & 0 & a & 0 & 0 \\ a & 0 & 0 & b & 0 & 0 \end{bmatrix}$$

Case III (*b*)

$$R_{abcd} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & -b & 0 & b & a \\ 0 & -b & -a & 0 & a & -b \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & a & 0 & -a & b \\ 0 & a & -b & 0 & b & a \end{bmatrix}$$

Cases I, II (b) and III (a) correspond to the types I, II and III of Petrov's classification. It has also been concluded by Sharma and Husain (1969) that cases III (a) and III (b) correspond to the gravitational radiation, and further that they are characterized by the following invariant conditions:

(i) A necessary and sufficient condition in order that R_{abcd} may belong to the case III (a) is that there exists a null vector l^a satisfying the conditions :

$$R_{abcd} l^a l^c = 0 \text{ and } R^*_{abcd} l^a l^c = 0.$$

where

$$R^*_{abcd} = \tfrac{1}{2} \eta_{abmn} R^{mn}_{cd}.$$

(ii) A necessary and sufficient condition in order that R_{abcd} may belong to the case III (b) is that there exists a null vector l^a satisfying the relations :

$$R_{abcd} l^a = 0 \text{ and } R^*_{abcd} l^a = 0.$$

(b) Géhéniau and Debever (1956) have decomposed the curvature tensor as

$$R_{abcd} = C_{abcd} + E_{abcd} + G_{abcd} \qquad \ldots(1)$$

where C_{abcd} is the Weyl tensor, E_{abcd} is the Einstein curvature tensor defined by

$$E_{abcd} = \tfrac{1}{2} (g_{ac} S_{bd} + g_{bd} S_{ac} - g_{ad} S_{bc} - g_{bc} S_{ad}) \qquad \ldots(2)$$

with

$$S_{ab} = R_{ab} - \tfrac{1}{4} g_{ab} R \qquad \ldots(3)$$

and

$$G_{abcd} = -\frac{R}{12} (g_{ac} g_{bd} - g_{ad} g_{bc}) \qquad \ldots(4)$$

Philip (1972) has enumerated fourteen invariants of the curvature tensor as follows :

There is a Ricci scaler R, there are four invariants of the Weyl tensor C_{abcd}, three invariants of the Einstein curvature tensor E_{abcd}, and six invariants of the combined Einstein and Weyl tensors.

From eqns. (2), (3) and (4), eqn. (1) may be expressed as

$$R_{abcd} = C_{abcd} + \tfrac{1}{2} (g_{ad} R_{bc} + g_{bc} R_{ad} - g_{ac} R_{bd} - g_{bd} R_{ac}) - \frac{R}{6} (g_{ad} g_{bc} - g_{ac} g_{bd}). \qquad \ldots(5)$$

If in the decomposition (5), the Ricci tensor $R_{ab} = 0$, then the Weyl tensor reduces to Riemann tensor and in this case, Riemann tensor has only four non-vanishing invariants. The component forms of these four invariants, as found out by Philip (1972), are as follows :

$$A_1 = R_{abcd}\, R^{abcd}; \quad A_2 = R^*_{abcd}\, R^{abcd}$$

$$B_1 = \tfrac{4}{3}\, R_{abcd}\, R^{cdmn}\, R^{ab}_{mn}\ ; \ B_2 = \tfrac{4}{3}\, R^*_{abc_d}\, R^{cdmn} R^{ab}_{mn}.$$

§2. *Forms of the Curvature Invariants in the Classification of Sharma and Husain* (1969) *and that of Petrov* — The forms of the invariants A_1, A_2, B_1 and B_2 of the curvature tensor as calculated for the classification of Sharma and Husain (1969) are summarized in Table I and those for Petrov's classification are given in Table II.

TABLE I

Forms of the invariants in Sharma-Husain classification

Case No.	A_1	A_2	B_1	B_2
I	$2(a_1^2 + a_2^2 + a_3^2 - b_1^2 - b_2^2 - b_3^2)$	0	$\frac{4}{3}[6(a_1b_1^2 + a_2b_2^2 + a_3b_3^2) - 2(a_1^3 + a_2^3 + a_3^3)]$	0
II (a)	$12(a_1^2 - b_1^2)$	0	$16a_1(a_1^2 - 3b_1^2)$	0
II (b)	$12(a^2 - b^2)$	0	$\frac{4}{3}(12a^3 - 36ab^2 - 32bcd)$	0
III (a)	0	0	0	0
III (b)	0	0	0	0

TABLE II

Forms of the invariants in Petrov classification

Case No.	A_1	A_2	B_1	B_2
I	$2(a_1^2 + a_2^2 + a_3^2) - 2(b_1^2 + b_2^2 + b_3^2)$	0	$\frac{4}{3}[6(a_1b_1^2 + a_2b_2^2 + a_3b_3^2) - 2(a_1^3 + a_2^3 + a_3^3)]$	0
II	$12(a^2 - b^2)$	0	$\frac{4}{3}(8a^3 - 36ab^2 + 12ac^2)$	0
III	0	0	0	0

From Tables I and II, we notice that A_1, A_2, B_1 and B_2 all are equal to zero for cases III (a) and III (b) or type III. Thus, we conclude that vanishing of the invariants of the curvature tensor in empty space-time corresponds to the case of gravitational radiation, or, in other words:

'If $R_{abcd} \neq 0$ and $A_1 = A_2 = B_1 = B_2 = 0$, then the gravitational radiation is present; otherwise there is no gravitational radiation.'

§3. Now, in order to check the validity of our assertion, we shall calculate the values of the four curvature invariants for some known solutions in general relativity, and compare the conclusions arrived at as a consequence of our assertion with the already known results about these solutions.

(i) Takeno's plane-wave solution (1961) is given by

$$ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - dz^2 + dt^2.$$

The non-vanishing components for the curvature tensor are

$$R_{3131} = -R_{3114} = R_{1414} = U$$
$$R_{2323} = -R_{2324} = R_{2424} = V$$
$$R_{3123} = -R_{3124} = -R_{1423} = R_{1424} = W.$$

On calculating the four curvature invariants, we get

$$A_1 = A_2 = B_1 = B_2 = 0$$

and hence, according to our assertion, Takeno's plane-wave solution corresponds to the case of gravitational radiation.

(ii) Einstein-Rosen metric (Rao 1963) is given by

$$ds^2 = e^{2\gamma-2\psi}(dt^2 - dr^2) - r^2e^{-2\psi}d\phi^2 - e^{2\psi}dz^2$$

where γ and ψ are functions of r and t only. We consider the case where

$$\psi = 0 \text{ and } \gamma = \gamma(r-t).$$

The non-vanishing components for the curvature tensor are

$$-R_{1212}/r^2 = -R_{1224}/r^2 = -R_{2424}/r^2 = R_{3131}$$
$$= -R_{3134} = R_{3434} = \gamma'/2r.$$

After calculations, we get

$$A_1 = A_2 = B_1 = B_2 = 0$$

and thus, according to our present assertion, this form of Einstein-Rosen metric belongs to the case of gravitational radiation.

(iii) The Peres metric (1959) is given by

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 - 2f(dx_4 + dx_3)^2 + dx_4^2.$$

The non-vanishing components of the curvature tensor are

$$R_{3131} = R_{3114} = R_{1414} = f_{,11}$$
$$R_{2323} = R_{2324} = R_{2424} = f_{,22}$$
$$R_{3123} = R_{3124} = R_{1423} = R_{1424} = f_{,12}.$$

After calculations, we get

$$A_1 = A_2 = B_1 = B_2 = 0$$

and thus, according to our present criterion, Peres line-element belongs to the case of gravitational radiation.

(iv) The line-element of Schwartzchild (see Tolman 1962) is given by

$$dx^2 = -e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + e^{\nu}\, dt^2$$

where $\nu = -\lambda$ and $e^{\nu} = \left(1 - \frac{2m}{r}\right)$ and λ, ν being functions of r. The non-vanishing components of the curvature tensor are

$$R_{1414} = (\tfrac{1}{2}\nu'' + \tfrac{1}{4}\nu'\lambda' - \tfrac{1}{4}\nu'^2)\, e^{\nu};\ R_{2424} = -\tfrac{1}{2} r\nu' e^{\nu-\lambda},$$
$$R_{3434} = -\tfrac{1}{2} r\nu' \sin^2\theta\, e^{\nu-\lambda};\ R_{2323} = r^2(e^{-\lambda} - 1)\sin^2\theta,$$
$$R_{3131} = -\tfrac{1}{2} r\lambda' \sin^2\theta;\ R_{1212} = -\tfrac{1}{2} r\lambda'.$$

After calculations, we find

$$A_1 = [(\tfrac{1}{2}\nu'' - \tfrac{1}{2}\nu'^2)^2 + (\nu'/r^2) + 1/r^4]\,(e^{2\nu}) - (2e^{-\lambda} + 1)/r^4,$$
$$A_2 = 0,$$
$$B_1 = \tfrac{4}{3}[\{(\tfrac{1}{2}\nu'' - \tfrac{1}{2}\nu'^2)^2(-\tfrac{1}{2}\nu'' + \tfrac{1}{2}\nu'^2) + \nu'^3/2r^3\}\,(e^{3\nu})$$
$$+ (e^{-\lambda} + 1)^3/r^6],$$
$$B_2 = 0,$$

which shows that, according to the present criterion, Schwartzchild line-element does not correspond to the case of gravitational radiation.

In fact, the matrix for R_{abcd} coincides with that of case II (b), if we put $b = c = d = 0$. Hence, Schwartzchild line-element belongs to the case II (b).

It has been shown by Sharma and Husain (1969) that Takeno's solution belongs to the case III (b) and corresponds to the gravitational radiation, whereas Schwartzchild line-element belongs to the case II (b) and does not correspond to gravitational radiation.

Further, it is well known (e.g. Rao 1963, Zakharov 1973) that both the Einstein-Rosen and Peres solutions belong to the type N and correspond to gravitational radiation.

Thus, we have verified that the conclusion as arrived at by our criterion tallies with the already known results.

§4. The 'super energy tensor' of Bel (1960) is defined by

$$T^{abcd} = \tfrac{1}{2}(R^{amcn} R^{bd}_{mn} + R^{*amcn} R^{*bd}_{mn})$$

and the 'super energy' of the gravitational field is defined by

$$W(u^a) = T^{abcd}u_a u_b u_c u_d$$

for any unit time-like vector u^a such that $u^2 = 1$. This $W(u^a)$ may be expressed as

$$W = \tfrac{1}{2}(Y_{ab}Y^{ab} + Z_{ab}Z^{ab})$$

where

$$Y_{ab} = R_{acbd}u^c u^d \text{ and } Z_{ab} = - R^*_{acbd}\, u^c u^d.$$

Calculating the super energy W for the different cases of the classification of Sharma and Husain (1969), we find that

For Case I ; $W = (a_1^2 + a_2^2 + a_3^2) - (b_1^2 + b_2^2 + b_3^2)$.

For Case II (a) ; $W = 6(a_1^2 - b_1^2)$.

For Case II (b) ; $W = 6(a^2 - b^2)$.

For Case III (a) ; $W = 0$.

For Case III (b) ; $W = 0$.

Thus, $W = 0$ for the cases III (a) and III (b) of the classification of Sharma and Husain (1969), which correspond to the gravitational radiation. However, it should be noted that, contrary to the theorem of Bel (1960), this will not imply that $R_{abcd} = 0$, because in these cases, there is no time-like vector u^a for which the relations

$$R_{abcd}u^b u^d = 0 \text{ and } R^*_{abcd}\, u^b u^d = 0$$

hold. In fact, the only vector for which these relations hold is the null vector. Thus, W becomes equal to zero without R_{abcd} vanishing, and, therefore, we can say that the gravitational radiation is characterized by the vanishing of the super energy of the gravitational field, while R_{abcd} is non-zero.

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