STRESSES IN SEMI-INFINITE STRIP UNDER SYMMETRIC LOAD ON THE LATERAL SIDES

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When stresses are prescribed on the two lateral sides of a semi-infinite strip, one side being stress-free, the way of calculating stresses at different points is shown. Approximation by Wiener-Hopf method has been employed.

INTRODUCTION

In the plane theory of elasticity, the problems for infinite strips have been a subject of interest for a long time. Howland (1929), Buchwald (1964) and Tiffen and Sample (1965) gave different approaches under different boundary conditions. Some authors employed Wiener-Hopf technique (cf. Matezyn'ski 1962). But the factorization process associated with this method is always involved. This is because of the fact that the roots of sinh $2au + 2au$ are complex and it is not possible to locate the positions of the roots accurately unless certain assumptions are made. White and Buchwald (1964) have factorized the above functions. But here also, the roots are not exactly located. The problem of semi-infinite strip is certainly harder than the problem of infinite strip. There are three boundaries instead of two, as in the case of an infinite strip. Nariboli (1965) made an attempt to solve the semi-infinite strip problem with orthogonality relations and eigen functions. But his method is not applicable for all types of boundary conditions. However, very little attention has been paid to this problem because of the obviously involved nature of the problem.

1. STATEMENT OF THE PROBLEM

The dimension of the strip is given by $x = \pm a$, $0 \leqslant y < \infty$. The y-axis is taken parallel to the length of the strip. The origin is at the middle of the other side. The boundary conditions are as follows:

$$\widehat{yy} - i\widehat{xy} = 0 \text{ at } y = 0 \qquad \ldots(1.1)$$

$$\widehat{xx} + i\widehat{xy} = f(y) \text{ at } x = \pm a, \text{ which is } O(1) \text{ as } y \to \infty \qquad \ldots(1.2)$$

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where $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ are usual notations of stresses. In the plane theory of elasticity, the stresses can be written in terms of complex potentials (cf. Muskhelishville 1953)

$$\widehat{xx} + \widehat{yy} = 2\{\Omega'(z) + \bar{\Omega}'(\bar{z})\}$$

$$\widehat{yy} - \widehat{xx} + 2i\,\widehat{xy} = 2\{\bar{z}\Omega''(z) + \omega''(z)\}$$

from which we can write

$$\widehat{xx} + i\,\widehat{xy} = \Omega'(z) + \bar{\Omega}'(\bar{z}) - z\bar{\Omega}''(\bar{z}) - \bar{\omega}''(\bar{z}) \qquad \text{...(1.3)}$$

$$\widehat{yy} - i\,\widehat{xy} = \Omega'(z) + \bar{\Omega}'(\bar{z}) + z\bar{\Omega}''(\bar{z}) + \bar{\omega}''(\bar{z}). \qquad \text{...(1.4)}$$

Since at $y = 0$, $\widehat{yy} - i\widehat{xy} = 0$, one of the unknown potentials, viz,, $\omega''(z)$, can be written in terms of the other by the method of analytic continuation with respect to the line $y = 0$, so that (1.3) and (1.4) are

$$\widehat{xx} + i\,\widehat{xy} = \Omega'(z) + 2\bar{\Omega}'(\bar{z}) + \Omega'(\bar{z}) - (z - \bar{z})\,\bar{\Omega}''(\bar{z}) \qquad \text{...(1.5)}$$

$$\widehat{yy} - i\,\widehat{xy} = \Omega'(z) - \Omega'(\bar{z}) + (z - \bar{z})\,\bar{\Omega}''(\bar{z}). \qquad \text{...(1.6)}$$

Obviously, (1.6) is identically satisfied at $y = 0$.

2. Solution of the Problem

We represent $\Omega'(z)$ as an integral

$$\Omega'(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{uz}\Lambda(u)\,du \qquad \text{...(2.1)}$$

where the integral is along the real line and $\Lambda(u)$ is a complex function of u. The boundedness of (2.1) is ensured by choosing the order of $\Lambda(u)$ suitably. Substituting the values of $\Omega'(z)$, etc., from (2.1) in eqn. (1.5), we get

$$\widehat{xx} + i\,\widehat{xy} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{uz}\Lambda(u) + 2e^{u\bar{z}}\,\bar{\Lambda}(u) + e^{u\bar{z}}\Lambda(u) - (z - \bar{z})\,e^{u\bar{z}}u\bar{\Lambda}(u)\right]du. \qquad \text{...(2.2)}$$

Using the boundary condition (1.2) at $x = a$ in eqn. (2.2), we get

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\Lambda(u)\,e^{iuy} + 2\bar{\Lambda}(u)\,e^{-iuy} + \Lambda(u)\,e^{-iuy} - 2iuy\bar{\Lambda}(u)\,e^{-iuy}\right]e^{au}\,du. \qquad \text{...(2.3)}$$

Now, we see

$$\int_{-\infty}^{\infty} \frac{d}{du}\left\{u\bar{\Lambda}(u)\, e^{au}\right\} e^{-iuy}du = \int_{-\infty}^{\infty} iuy\bar{\Lambda}(u)\, e^{au}e^{-iuy}du \quad \text{...(2.4)}$$

assuming $\bar{\Lambda}(u) \sim O(e^{-a|u|}u^{-1})$, the term within bracket vanishes on both limits. Hence, using (2.4) in (2.3), we get

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\Lambda(-u)\, e^{-au} + 2\bar{\Lambda}(u)\, e^{au} + \Lambda(u)\, e^{au} - 2\frac{d}{du}\left\{u\bar{\Lambda}(u)\, e^{au}\right\}\Big] e^{-iuy}\, du$$

which is equivalent to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\Lambda(-u)\, e^{-au} + \Lambda(u)\, e^{au} - 2au\bar{\Lambda}(u)\, e^{au} - 2u\bar{\Lambda}'(u)\, e^{au}\Big] e^{-iuy}\, du \quad \text{...(2.5)}$$

The relation (2.5) is in the form of Fourier integral, which is valid for $y \geqslant 0$. Then by the inversion formula of complex Fourier transform

$$\Lambda(-u)\, e^{-au} + \{\Lambda(u) - 2au\bar{\Lambda}(u) - 2u\bar{\Lambda}'(u)\}\, e^{au} = F_+(u) + G_-(u) \quad \text{...(2.6)}$$

where

$$F_+(u) = \frac{1}{2\pi}\int_0^{\infty} f(y)\, e^{iuy}\, dy \text{ and } G_-(u) = \frac{1}{2\pi}\int_{-\infty}^{0} g(y)\, e^{iuy}\, dy. \quad \text{...(2.7)}$$

Evidently, $F_+(u)$ is known and $G_-(u)$ is unknown. In a similar manner, we get another equation for the boundary condition at $x = -a$ as

$$\Lambda(-u)\, e^{au} + \{\Lambda(u) + 2au\bar{\Lambda}(u) - 2u\bar{\Lambda}'(u)\}\, e^{au} = F_+(u) + G_-(u) \quad \text{...(2.8)}$$

where it is assumed that unknown integral $G-(u)$ for both eqns. (2.6) and (2.8) are the same. Multiplying (2.6) by e^{-au} and (2.8) by e^{au} and subtracting, we get

$$\sinh 2au\Lambda(-u) + 2au\bar{\Lambda}(u) = \{F_+(u) + G_-(u)\}\sinh au. \quad \text{...(2.9)}$$

Taking complex conjugate and changing u to $-u$ of both sides of eqn. (2.9), we get

$$\sinh 2au\bar{\Lambda}(u) - 2au\Lambda(-u) = \{\bar{F}_+(-u) + \bar{G}_-(-u)\}\sinh au. \quad \text{...(2.10)}$$

If for simplicity we assume that $f(y)$ and $g(y)$ are purely real

i.e., $\widehat{xy} = 0$ on $x = \pm a$, $F_+(u) = \bar{F}_+(-u)$; $G_-(u) = \bar{G}_-(-u)$

then eqn. (2.10) becomes

$$\sinh 2au\bar{\Lambda}(u) + 2au\Lambda(-u) = \{F_+(u) + G_-(u)\} \sinh au. \qquad ...(2.11)$$

From (2.9) and (2.11), we obtain the values of $\bar{\Lambda}(u)$ and $\Lambda(-u)$ in terms of $F_+(u)$ and $G_-(u)$. These are given by

$$\bar{\Lambda}(u) = \Lambda(-u) = \frac{F_+(u) + G_-(u)}{\sinh 2au + 2au} \sinh au. \qquad ...(2.12)$$

Then, we multiply (2.6) by e^{au} and (2.8) by e^{-au}. On subtracting, we get another equation similar to (2.9), viz.,

$$\sinh 2au \{\Lambda(u) - 2u\bar{\Lambda}'(u)\} - 2au \cosh 2au\bar{\Lambda}(u) = \{F_+(u) + G_-(u)\} \sinh au. \qquad ...(2.13)$$

We now substitute the values of $\bar{\Lambda}(u)$, $\Lambda(-u)$, $\Lambda(u)$ and $\bar{\Lambda}'(u)$ from (2.12) in (2.13) and get the following equation of Wiener-Hopf type:

$$\{F_+(-u) + G_-(-u)\} - 2u\{F'_+(u) + G'_-(u)\} - \{K(u) + 1\} \times \{F_+(u) + G_-(u)\} = 0 \qquad ...(2.14)$$

where $K(u)$ is given by

$$K(u) = \frac{8a^2u^2 \coth au}{\sinh 2au + 2au}. \qquad ...(2.15)$$

3. Solution of Wiener-Hopf Equation

The common strip of regularity of eqn. (2.14) can be found out from the conditions imposed upon $F_+(u)$ and $G_-(u)$ for their convergence. We proceed as follows:

Let us assume that $f(y) \sim O(e^{\epsilon_1 y})$ and $g(y) \sim O(e^{\epsilon_2 y})$, where ϵ_1 and ϵ_2 are arbitrary small positive real numbers. Then, $F_+(u)$ is regular and convergent for $\text{Im}(u) > \epsilon_1$ and $G_-(u)$ is regular and convergent for $\text{Im}(u) > \epsilon_2$. It is seen from eqn. (2.7) that $F_+(u)$, $G_-(u) \sim O(|u|^{-1})$ as $u \to \infty$ in appropriate half planes. Taking $\epsilon_1 \gg \epsilon_2$, the strip where eqn. (2.14) is regular is given by $\epsilon_1 < \text{Im}(u) < \epsilon_2$. It is seen that the standard Wiener-Hopf technique can be applied now. But to proceed further, we see that complete factorization of $K(u)$ is not possible because of the term $\sinh 2au + 2au$ in the denominator of $K(u)$ which cannot be factorized in a simple way (cf. Buchwald 1964). To overcome this difficulty, we follow the method suggested by Koiter outlined in Noble (1958). To approximate $K(u)$ by another function $K^*(u)$, we take

$$K^*(u) = \frac{2\sqrt{3}\, K_1 u \cosh K_1 u}{\sinh \sqrt{3}\, K_1 u} \qquad ...(3.1)$$

where $K_1 = 1.2778\, a.K^*(u)$ is found out from $K(u)$ by comparing term by term up to 4th order. It is easy to check whether $K(u)$ and $K^*(u)$ are approximately equal, since it is necessary only to compare the numerical values of these functions on some line, $\operatorname{Im}(u) = u_1, -\infty < \operatorname{Re}(u) < \infty$, where $\epsilon_1 < u_1 < \epsilon_2$. A check table is given in the appendix.

Since $F_+(u)$ is known, $K^*(u)\, F_+(u) = L_+(u) + L_-(u)$, where $L_+(u)$ and $L_-(u)$ are regular in the appropriate half planes. $K^*(u)\, G_-(u)$ is resolved as follows: $G_-(u)$ is analytic in the lower half and $K^*(u)$ has singularity in both upper and lower halves. Therefore, $K^*(u)\, G_-(u) = M_+(u) + M_-(u)$, where

$$M_+(u) = \sum_{s=1}^{\infty} \frac{K^*(-i\beta_s)\, G_-(-i\beta_s)}{u + i\beta_s} \qquad \ldots(3.2)$$

$$M_-(u) = K^*(u)\, G_-(u) - M_+(u). \qquad \ldots(3.3)$$

The poles of $K^*(u)$ in the lower half plane are $-i\beta_s$,

where $$\beta_s = \frac{s\pi}{\sqrt{3}\, K_1},\ s = 1, 2, \ldots. \qquad \ldots(3.4)$$

Hence, eqn. (2.14) can be written as

$$G_-(-u) - 2uF'_+(u) - F_+(u) - L_+(u) - M_+(u)$$

$$= -F_+(-u) + 2uG'_-(u) + G_-(u) + L_-(u) + M_-(u) \qquad \ldots(3.5)$$

where left and right sides are regular and tend to zero as $u \to \infty$ in appropriate half-planes. Hence, on applying Liouville's theorem, each side of the equation equals zero. Hence,

$$G_-(-u) - 2uF'_+(u) - F_+(u) - L_+(u) - M_+(u) = 0 \qquad \ldots(3.6)$$

4. A Particular Example

We have to assume some particular value of $f(y)$. For simplicity, we take $f(y) = 1,\ 0 \leqslant y < \infty$, so that

$$F_+(u) + 2uF'_+(u) = \frac{1}{iu}\cdot \qquad \ldots(4.1)$$

$L_+(u)$, which is a part of $K^*(u)\, F_+(u)$ in the upper half plane, can be obtained using the well-known result

$$\frac{\cos \alpha z}{\sin \pi z} = \frac{1}{\pi z} + \frac{2z}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\cos n\alpha}{z^2 - n^2},\ -\pi < \alpha < \pi. \qquad \ldots(4.2)$$

If $\alpha = \pi/\sqrt{3}$ and $z = \sqrt{3}K_1 iu/\pi$, then $K^*(u)\,F_+(u)$ can be written as

$$K^*(u)\,F_+(u) = 2i\left[\frac{1}{u} + 2u\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{n\pi}{\sqrt{3}}\right)}{u^2+\frac{n^2\pi^2}{3K_1^2}}\right] = L_+(u) + L_-(u) \qquad \ldots(4.3)$$

where
$$L_+(u) = 2i\sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{n\pi}{\sqrt{3}}\right)\left(\frac{1}{u+\frac{n\pi i}{\sqrt{3}K_1}} - \frac{1}{\frac{n\pi i}{\sqrt{3}K_1}}\right) \qquad \ldots(4.4)$$

and

$$L_-(u) = 2i\left[\frac{1}{u} + \sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{n\pi}{\sqrt{3}}\right)\left(\frac{1}{u-\frac{n\pi i}{\sqrt{3}\,K_1}} + \frac{1}{\frac{n\pi i}{\sqrt{3}\,K_1}}\right)\right]. \qquad \ldots(4.5)$$

Therefore, substituting the values from (3.2), (3.4), (4.1) and (4.4) in eqn. (3.6), we get

$$G_-(-u) + \frac{i}{u} + 2i\sum_{s=1}^{\infty}\frac{(-1)^s\frac{s\pi}{\sqrt{3}\,K_1}\cos\left(\frac{s\pi}{\sqrt{3}\,K_1}\right)G_-\left(-\frac{s\pi i}{\sqrt{3}\,K_1}\right)}{u+\frac{s\pi i}{\sqrt{3}\,K_1}}$$

$$= 2i\sum_{s=1}^{\infty}(-1)^s\cos\left(\frac{s\pi}{\sqrt{3}}\right)\left(\frac{1}{u+\frac{s\pi i}{\sqrt{3}\,K_1}} - \frac{1}{\frac{s\pi i}{\sqrt{3}\,K_1}}\right). \qquad \ldots(4.6)$$

An infinite set of simultaneous linear algebraic equations is obtained by setting $u = i\beta_r = \frac{r\pi i}{\sqrt{3}\,K_1}$ in (4.6), viz.,

$$x_r + \frac{\sqrt{3}\,K_1}{r\pi} + 2\sum_{s=1}^{\infty}\frac{(-1)^s\,s\cos\left(\frac{s\pi}{\sqrt{3}}\right)x_s}{r+s}$$

$$= \frac{2\sqrt{3}\,K_1}{\pi}\sum_{s=1}^{\infty}(-1)^s\cos\left(\frac{s\pi}{\sqrt{3}}\right)\left(\frac{1}{r+s} - \frac{1}{s}\right) \qquad \ldots(4.7)$$

where

$$x_r = G_-\left(-\frac{r\pi i}{\sqrt{3}K_1}\right), \quad r = 1, 2, \ldots$$

5. Numerical Calculations

The first four equations in (4.7), assuming x_s for $s > 4$ are

$$1.2406\, x_1 + 1.1123\, x_2 - 0.4996\, x_3 + 0.9018\, x_4 = -\ 0.6279$$

$$0.1604\, x_1 + 0.1658\, x_2 - 0.3997\, x_3 + 0.7515\, x_4 = -\ 0.2080$$

$$0.1203\, x_1 + 0.6674\, x_2 + 0.6669\, x_3 + 0.6441\, x_4 = -\ 0.0442$$

$$0.0962\, x_1 + 0.5561\, x_2 - 0.2855\, x_3 + 1.5636\, x_4 = \ \ 0.0873$$

Solving the above equations, the values of x_1, x_2, x_3 and x_4 are 0.9679; 0.4811; 0.2695; and 0.1073.

Substituting the values of x_1, x_2, x_3, etc., $G_-(-u)$ is obtained from (4.6) and consequently $\Lambda(u)$ is obtained from (2.12), which in turn gives the stress components from (2.2).

Values of horizontal $(\widehat{xx})$ and tangential $(\widehat{xy})$ stresses calculated for fixed x and y are given in Table I

TABLE I

Values of $\widehat{xx}$ *and* $\widehat{xy}$

y	x	$\widehat{xx}$	$\widehat{xy}$
0.25	0	0.621	0.038
	0.25	0.597	0.014
	0.50	0.603	− 0.014
	0.75	0.667	− 0.056
	1.0	0.851	− 0.136
1.0	0.0	0.971	0.216
	0.25	0.905	0.136
	0.50	0.825	0.071
	0.75	0.694	0.050
	1.0	0.412	0.188
2.0	0.0	0.955	0.403
	0.25	0.941	0.335
	0.50	0.889	0.275
	0.75	0.809	0.225
	1.0	0.553	0.202

(*continued on p.* 670)

TABLE I (*contd.*)

3.0	0.0	0.777	0.402
	0.25	0.792	0.353
	0.50	0.789	0.300
	0.75	0.735	0.226
	1.0	0.485	0.062
4.0	0.0	0.620	0.268
	0.25	0.654	0.251
	0.50	0.671	0.217
	0.75	0.646	0.122
	1.0	0.479	− 0.218

Remarks: We may get results with greater precision by making better approximation of $G_{-}(-u)$ in (4.6), which can be done with the help of (4.7) by taking larger values of s.

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APPENDIX

(A)
$$K(u) = 2\left(1 - \frac{4}{45}a^4u^4 - \dots\right) \quad \dots(a)$$

$$K^*(u) = \frac{K_2\left(1 + \frac{K_1^2u^2}{2} + \frac{K_1^4u^4}{24} + \dots\right)}{K_3\left(1 + \frac{K_3^2u^2}{6} + \frac{K_3^4u^4}{120} + \dots\right)} \quad \dots(b)$$

where the values of K_1, K_2, K_3, will be chosen suitably.

Put $K_3 = \sqrt{3}\,K_1$, $K_2 = 2\sqrt{3}\,K_1$, then

$$K^*(u) = 2\left(1 - \frac{K_1^4 u^4}{3} - \ldots\right). \qquad \ldots(c)$$

Comparing (a) and (c) term by term, we get $K_1 = \sqrt[3]{\frac{8}{3}}\,a$.

(B) $a = 1$

u	0.2	0.5	1.0	1.5	2.0
$K(u)$	2.0000	1.9896	1.8668	1.5276	1.0608
$K^*(u)$	2.0003	1.9902	1.8946	1.6828	1.4052
$K^* - K$	0.0003	0.0006	0.0278	0.1552	0.3444