SOME THEOREMS OF AFFINE MOTION IN A RECURRENT FINSLER SPACE IV

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In this paper the author has continued his study on the affine motion characterized by the equations A, B and C in a recurrent Finsler space. All the above three types of affine motion have been discussed separately and many important results have been obtained.

1. INTRODUCTION

Let F_n (Rund 1959) be an n-dimensional affinely connected and non-flat Finsler space having a fundamental metric function $F(x, \dot{x})$ and a symmetric connection coefficient $\Gamma^{*i}_{jk}(x, \dot{x})$. The covariant derivative of tensor field $T^i_j(x, \dot{x})$ with respect to x^k in the sense of Cartan is given by

$$T^i_{j|k} = \partial_k T^i_j - \dot{\partial}_s T^i_j G^s_k + T^s_j \Gamma^{*i}_{sk} - T^i_s \Gamma^{*s}_{jk} \cdot \qquad \ldots(1.1)$$

The commutation formula involving the above covariant derivative is given by

$$2 T^{i(1)}_{j|[hk]} = - \dot{\partial}^{(2)}_r T^i_j K^r_{shk} \dot{x}^s - T^i_s K^s_{jhk} + T^s_j K^i_{shk} \qquad \ldots(1.2)$$

where

$$K^i_{hjk}(x, \dot{x}) \overset{def}{=} 2 \{\partial_{[k} \Gamma^{*i}_{j]h} - \dot{\partial}_r \Gamma^{*i}_{h[j} G^r_{k]} + \Gamma^{*r}_{h[j} \Gamma^{*i}_{k]r}\}. \qquad \ldots(1.3)$$

is called Cartan's curvature tensor field and satisfies the following relations (Rund 1959) :

$$K^i_{hjk} + K^i_{jkh} + K^i_{khj} = 0 \qquad \ldots(1.4)$$

$$K^i_{hjk} = - K^i_{hkj} \qquad \ldots(1.5)$$

and

$$K^i_{hji} = K_{hj}. \qquad \ldots(1.6)$$

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(1) $2A_{[hk]} = A_{hk} - A_{kh}$.

(2) $\dot{\partial}_i \equiv \partial/\partial\dot{x}^i$ and $\partial_i \equiv \partial/\partial x^i$

Let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\, dt \qquad \text{...(1.7)}$$

where $v^i(x)$ is any vector field and dt is an infinitesimal point constant.

In view of the above covariant derivative and point transformation, the Lie-derivatives of $T^i_j(x, \dot{x})$ and $\Gamma^{*i}_{jk}(x, \dot{x})$ are given by (Yano 1957) :

$$\mathcal{L}v\, T^i_j = T^i_{j|h}\, v^h - T^h_j\, v^i_{|h} + T^i_h\, v^h_{|j} + \dot{\partial}_h T^i_j\, v^h_{|\gamma}\, \dot{x}^\gamma$$

and

$$\mathcal{L}v\, \Gamma^{*i}_{jk} = v^i_{|jk} - K^i_{jkh}\, v^h + \dot{\partial}_s \Gamma^{*i}_{jk}\, v^s_{|\gamma}\, \dot{x}^\gamma. \qquad \text{...(1.8)}$$

We have also the following well-known commutation formula

$$\mathcal{L}v\,(T^i_{j|h}) - (\mathcal{L}v\, T^i_j)_{|h} = 0. \qquad \text{...(1.9)}$$

In an n-dimensional space, if the curvature tensor $K^i_{hjk}(x, \dot{x})$ satisfies the following relation :

$$K^i_{hjk|s} = \gamma_s K^i_{hjk}, \qquad \text{...(1.10)}$$

where $\gamma_s(x)$ is any covariant vector field, then the space is called a recurrent Finsler space of first order or F^*_n-space.

In an F^*_n- space, the gradient vector is given by

$$\gamma_h \equiv \frac{1}{\gamma}\, \partial_h \gamma. \qquad \text{...(1.11)}$$

In an earlier paper, we have concluded as follows : If an F^*_n-space admits an infinitesimal affine motion $\bar{x}^i = x^i + v^i(x)\, dt$, we have

$$\gamma_h v^h = 0 \qquad \text{...(1.12)}$$

say, the function $\gamma(x)$ being a Lie-invariant one.

Now, let us operate $\mathcal{L}v$ to both sides of the fundamental and starting condition (1.10), we find

$$(\mathcal{L}v K^i_{hjk})_{|s} = (\mathcal{L}v \gamma_s)\, K^i_{hjk} + \gamma_s \mathcal{L}v\, K^i_{hjk}\,. \qquad \text{...(1.13)}$$

If the F^*_n-space admits an affine motion, then we have

(a) $\mathcal{L}v K^i_{hjk} = 0$

and

(b) $\mathcal{L}v \Gamma^{*i}_{jk} = v^i_{|jk} - K^i_{jkh} v^h + \dot{\partial}_h \Gamma^{*i}_{jk} v^h_{|\gamma} \dot{x}^\gamma = 0.$ ...(1.14)

Therefore, in view of (1.14), eqn. (1.13) may be written as

$$\mathcal{L}v \gamma_s K^i_{hjk} = 0. \quad ..(1.15)$$

Since the space is non-flat (i.e., $K^i_{hjk} \neq 0$), eqn. (1.15) yields

$$\mathcal{L}v \gamma_s = 0. \quad ...(1.16)$$

From the above relation, however, in the present gradient case, we obtain

$$\gamma_{s|h} = \gamma_{h|s}. \quad ...(1.17)$$

Equation (1.17) can also be written as

$$(\gamma_s v^s)_{|h} = 0. \quad ...(1.18)$$

$$\gamma_s v^s = c \quad ...(1.19)$$

where c is an arbitrary constant.

2. Some Appendices to the Recurrent Motion

The present author has discussed earlier the affine motion of recurrent form (Kumar 1976b). In the same paper the concrete form of such motion was pursued and the following two cases were obtained :

(a) $\gamma_s + \phi_s = 0$

and

(b) $K_{hj} v^j = 0.$ .. (2.1)

In what follows only case (2.1a) has been taken up.

In fact, differentiating (2.1b) covariantly, we can get

$$(\gamma_s + \phi_s) K_{hj} v^j = 0. \quad ...(2.2)$$

Therefore, $(\gamma_s + \phi_s)$ may be taken arbitrarily. In this meaning, the former case means only a special case contained in (2.1b). Thus, we know that in order to discuss generally the recurrent affine motion in an F^*_n, we must take up the case where (2.1b) holds good. We consider below this general case.

By virtue of eqns. (1.6) and (2.1b), we can have

$$K^i_{hji}\, v^j = 0. \qquad \ldots(2.3)$$

Now, contracting the well-known Bianchi's identity (1.4) for Cartan's curvature tensor with respect to i and k, we get

$$K^i_{hji} = K^i_{jih} - K^i_{ihj}\,. \qquad \ldots(2.4)$$

Transvecting the last formula by v^j and using eqn. (2.3), we obtain

$$K^i_{jih}\, v^j + K^i_{ihj}\, v^j = 0. \qquad \ldots(2.5)$$

In view of eqns. (1.5) and (1.14b), eqn. (2.5) takes the form

$$v^i_{\,|\,ij} - K^i_{hji}\, v^h = 0. \qquad \ldots(2.6a)$$

Since $\quad v^i_{\,|\,h} = \phi_h v^i$, we have

$$v^i_{\,|\,ij} = (\phi_i v^i)_{\,|\,j}. \qquad (2.6b)$$

Thus, by virtue of the above formula, eqn. (2.6) can be written as

$$K_{jh}\, v^j = (\phi_s\; v^s)_{\,|\,h}. \qquad \ldots(2.7)$$

Differentiating the above formula covariantly with respect to x^m and noting (1.10) and

$$v^i_{\,|\,h} = \phi_h v^i \qquad \ldots(2.8)$$

we can easily obtain

$$(\gamma_m + \phi_m)\, K_{jh} v^j = (\phi_s v^s)_{\,|\,hm}. \qquad \ldots(2.9)$$

Eliminating the term $K_{hj} v^j$ with the help of (2.7) and (2.9), we find

$$(\gamma_m + \phi_m)\, (\phi_s v^s)_{\,|\,h} = (\phi_s v^s)_{\,|\,hm}. \qquad \ldots(2.10)$$

Because of the symmetry of connection $\Gamma^{*i}_{jk}\,(x, \dot{x})$, we can find

$$(\phi_s v^s)_{\,|\,hm} = (\phi_s v^s)_{\,|\,mh}. \qquad \ldots(2.11)$$

Consequently, with the help of (2.10) and (2.11), we have

$$(\gamma_h + \phi_h)\, (\phi_s v^s)_{\,|\,k} = (\gamma_k + \phi_k)\, (\phi_s v^s)_{\,|\,h}. \qquad \ldots(2.12)$$

In view of eqns. (2.1*b*) and (2.7), we can construct

$$(\phi_s v^s)_{\,|\,j} v^j = (K_{mj} v^m) v^j = (K_{mj} v^j) v^m = 0. \qquad \ldots(2.13)$$

Therefore, transvecting (2.12) by v^k and taking note of the above aquation, we get

$$(\gamma_k + \phi_k) v^k\, (\phi_s v^s)_{\,|\,h} = 0. \qquad \ldots(2.14)$$

Thus, we obtain here two cases :

(a) $(\gamma_k + \phi_k)v^k = 0$

or

(b) $\phi_s v^s = \text{const.}$ ...(2.15)

In this way, from (2.15), we can say that if an F_n^*-space admits an affine motion of recurrent form, then there exist the following two interesting fields :

(i) A case of $(\gamma_h + \phi_h)v^h = 0$

(ii) A case of $\phi_h v^h = \text{const.}$ ...(2.16)

If we take especially $\phi_h = -\gamma_h$ so as to satisfy (2.15*a*), we can develop the existence theory (Kumar 1976b). Thus, we know that there is the possibility of the existence of three categories of recurrent affine motion, viz.

(A) $\bar{x}^i = x^i + v^i(x)\, dt,\ v^i_{\ |j} = \phi_j v^i,\ \phi_h v^h = \text{const.},$

(B) $\bar{x}^i = x^i + v^i(x)\, dt,\ v^i_{\ |j} = \phi_j v^i,\ (\phi_h + \gamma_h)\, v^h = 0,$

(C) $\bar{x}^i = x^i + v^i(x)\, dt,\ v^i_{\ |j} = \phi_j v^i,\ \phi_h + \gamma_h = 0$

being derived from (B) formally. In section 5, an attempt has been made to clarify the concrete relation existing between (B) and (C).

3. Study of Case (A)

1. *Necessary Conditions*

In this case, we have

$$\phi_h v^h = \text{const.} \quad ...(3.1)$$

Differentiating covariantly (3.1) by x^s and taking note of the condition (A), we find

$$\phi_{h|s} v^h + \phi_h \phi_s v^h = 0. \quad ...(3.2)$$

By virtue of (1.14*b*) and the condition (A), we find

$$K^i_{jkh}\, v^h = (\phi_j v^i)_{|k} = \phi_{j|k} v^i + \phi_j \phi_k v^i. \quad ...(3.3)$$

Transvecting the last formula by v^k and taking note of the fact that $K^i_{jkh}\, v^k v^h = 0$, we get

$$\phi_{j|k} v^k + \phi_j \phi_k v^k = 0. \quad ...(3.4)$$

Now, comparing (3.2) and (3.4), we can obtain

$$(\phi_{j|k} - \phi_{k|j})\, v^k = 0. \quad ...(3.5)$$

In an affinely connected space, the second Bianchi's identity can be written as

$$K^i_{hjk|s} + K^i_{hks|j} + K^i_{hsj|k} = 0 \qquad ...(3.6)$$

which by virtue of definition (1.10) takes the form

$$K^i_{hjk} v^k \gamma_s + K^i_{hks} v^k \gamma_j + K^i_{hsj} \gamma_k v^k = 0. \qquad ...(3.7)$$

Making a cyclic interchange with respect to the indices h, s, j in the above formula and taking notice of (1.4) and (1.5), we have

$$(K^i_{hjk} v^k - K^i_{jhk} v^k) \gamma_s + (K^i_{shk} v^k - K^i_{hsk} v^k) \gamma_j + (K^i_{jsk} v^k - K^i_{sjk} v^k) \gamma_h = 0. \qquad ...(3.8)$$

In view of (1.14*b*), (3.8) may be replaced by

$$(v^i_{|hj} - v^i_{|jh}) \gamma_s + (v^i_{|sh} - v^i_{|hs}) \gamma_j + (v^i_{|js} - v^i_{sj}) \gamma_h = 0. \qquad ...(3.9)$$

By virtue of the condition (A), eqn. (3.9) takes the form

$$(\phi_{h|j} - \phi_{j|h}) v^i \gamma_s + (\phi_{s|h} - \phi_{h|s}) v^i \gamma_j + (\phi_{j|s} - \phi_{s|j}) v^i \gamma_h = 0. \qquad ...(3.10)$$

Contracting eqn. (3.10) with respect to indices i and h and making use of enq. (3.5), we get

$$v^h \gamma_h (\phi_{j|s} - \phi_{s|j}) = 0. \qquad ...(3.11)$$

Equation (3.11) gives two conditions :

(i) $v^h \gamma_h = 0$

or

(ii) $\phi_{j|s} = \phi_{s|j}$. ...(3.12)

Hence, we can state that when an F^*_n-space admits an affine motion of recurrent form, a condition $\phi_h v^h$ = constant is necessitated. But in such a case, we have the following :

(A) $v^h \gamma_h = 0$ should be satisfied.

(B) Defining vector ϕ_j of the motion should be gradient vector.

2. *The case of* ϕ_j = *gradient vector*

In such a case, the motion takes the following form :

$$\bar{x}^i = x^i + v^i(x)\, dt, \ v^i_{|j} = \phi_j v^i, \ \phi_j(x) = \text{gradient vector}. \qquad ...(3.13)$$

Let us define the following commutator

$$\rho^i_{jk} \equiv (v^i_{\ |jk} - v^i_{\ |kj}). \qquad ...(3.14)$$

In view of (3.13), the above formula can be written as

$$\rho^i_{jk} = \phi_{j|k} v^i + \phi_j v^i_{\ |k} - \phi_{k|j} v^i + \phi_k v^i_{\ |j} = 0 \qquad ...(3.15)$$

where we have used the gradient property of ϕ_j :

$$\phi_{j|k} = \phi_{k|j}. \qquad ...(3.16)$$

On the other hand, by virtue of the commutation formula (1.2) and (3.15), the relation (3.14) can be written as

$$v^h K^i_{hjk} = 0. \qquad ...(3.17)$$

By the general theory of fields of parallel vectors (Eisenhart 1927), (3.17) shows that v^i determines a field of parallel vectors. Consequently, the motion (3.13) under consideration is not a pure recurrent motion, but a contra-motion in the general sense. Now, if (3.13) denotes exactly an affine motion, it has to satisfy the integrability condition.

$$\mathcal{L}v\, K^i_{hjk} = K^i_{hjk} \gamma_s v^s - K^s_{hjk}\, v^i_{\ |s} + K^i_{sjk}\, v^s_{\ |h} + K^i_{hsk}\, v^s_{\ |j}$$

$$+ K^i_{hjs}\, v^s_{\ |k} + \dot{\partial}_s K^i_{hjk}\, v^s_{\ |\gamma}\, \dot{x}^\gamma = 0. \qquad ...(3.18)$$

of the equation of affine motion (1.14*b*).

On substituting the value of $v^i_{\ |j}$ from (3.13) into the last formula, we obtain

$$\gamma_s v^s K^i_{hjk} - K^s_{hjk}\, \phi_s v^i + K^i_{sjk}\, \phi_h v^s + K^i_{hsk}\, \phi_j v^s + K^i_{hjs}\, \phi_k v^s = 0. \qquad ...(3.19)$$

But in view of (3.17), the above relation may be replaced by

$$\gamma_s v^s K^i_{hjk} - K^s_{hjk}\, \phi_s v^i + K^i_{hsk}\, \phi_j v^s + K^i_{hjs}\, \phi_k v^s = 0. \qquad ...(3.20)$$

Contracting eqn. (20) with respect to indices i and k, we find

$$\gamma_s v^s K_{hj} + \phi_j K_{hs} v^s = 0. \qquad ...(3.21)$$

The present theory is based on the condition (2.1*b*) and being $\gamma_s v^s \neq 0$, (3.21) yields

$$K_{hj} = 0. \qquad ...(3.22)$$

From the above discussion, we come to the following conclusion.

Theorem 3.1 — If an F_n^*-space admits a special recurrent motion or contra-motion in the general sense of the form derived from (A) :

$$\bar{x}^i = x^i + v^i(x)\, dt,\ v^i_{\ |j} = \phi_j v^i,\ \phi_{j|k} = \phi_{k|j}, \qquad \text{...(3.23)}$$

with $\gamma_s v^s \neq 0$, the space has the property represented by (3.22).

4. Study of Case (B)

For this case, the starting assumption is

$$(\phi_h + \gamma_h) v^h = 0. \qquad \text{...(4.1)}$$

Transvecting the integrability condition (3.19) of affine motion by v^k and using condition B and the last formula, we find

$$\begin{aligned} &- \phi_s v^s(\phi_{h|j} v^i + \phi_h \phi_j v^i) - \phi_s v^i(\phi_{h|j} v^s + \phi_h \phi_j v^s) \\ &+ \phi_h v^s(\phi_{s|j} v^i + \phi_s \phi_j v^i) + \phi_j v^s(\phi_{h|s} v^i + \phi_h \phi_s v^i) \\ &+ \phi_k v^k(\phi_{h|j} v^i + \phi_h \phi_j v^i) = 0 \end{aligned} \qquad \text{...(4.2)}$$

where we have also used

$$K^i_{hjk}\, v^k = \phi_{|hj} - \dot{\partial}_s \Gamma^{*i}_{hj}\, v^s_{\ |\gamma}\, \dot{x}^\gamma = (\phi_h v^i)_{|j} = \phi_{h|j} v^i + \phi_h \phi_j v^i. \qquad \text{...(4.3)}$$

After a little simplification, (4.2) can be written as

$$- \phi_s v^s \phi_{h|j} v^i + \phi_h v^s \phi_{s|j} v^i + \phi_j v^s \phi_{h|s} v^i + \phi_k \phi_h \phi_j v^k v^i = 0. \qquad \text{...(4.4)}$$

Commutating eqn. (4.4) with respect to indices h and j, we have

$$- \phi_s v^s(\phi_{h|j} - \phi_{j|h}) + \phi_h(\phi_{s|j} - \phi_{j|s}) + \phi_j(\phi_{h|s} - \phi_{s|h})\, v^s = 0 \qquad \text{...(4.5)}$$

where we have neglected non-null v^i.

Contracting (3.10) with respect to indices i and s, and taking the case of (4.1), we get

$$- \phi_s v^s(\phi_{h|j} - \phi_{j|h}) - \gamma_j(\phi_{h|s} - \phi_{s|h})\, v^s - \gamma_h(\phi_{s|j} - \phi_{j|s})\, v^s = 0. \qquad \text{...(4.6)}$$

Subtracting (4.6) from (4.5), we obtain

$$(\phi_j + \gamma_j)(\phi_{h|s} - \phi_{s|h})\, v^s + (\phi_h + \gamma_h)(\phi_{s|j} - \phi_{j|s})\, v^s = 0. \qquad \text{...(4.7)}$$

Putting

$$M_s \equiv (\phi_{h|s} - \phi_{s|h})\, v^h \qquad \text{...(4.8)}$$

when and only when $M_s \neq 0$, there exists a suitable proportional scalar function $\beta(x)$, such that

$$\phi_s + \gamma_s = \beta(x) M_s \qquad \text{...(4.9)}$$

where M_s satisfies the condition

$$M_s v^s = 0. \qquad \ldots(4.10)$$

By virtue of (4.8) and the basic condition $K_{hj} v^j = 0$, we have

$$M_s v^s = \{(\phi_h v^h)_{|s} - \phi_h v^h_{|s} - (\phi_s v^h)_{|h} + \phi_s v^h_{|h}\} v^s$$

$$= \{v^h_{|hs} - \phi_h \phi_s v^h - v^h_{|sh} + \phi_s \phi_h v^h\} v^s$$

$$= \{v^h_{|hs} - v^h_{|sh}\} v^s = (v^m K^h_{msh}) v^s = (K_{ms} v^m) v^s$$

$$= (K_{ms} v^s) v^m = 0 \qquad \ldots(4.11)$$

where we have also used the commutation formula (1.2) and (1.6). This completes the proof of the condition (4.10). Being M_s given as we have mentioned above, by

$$M_s = K^h_{msh} v^m = K_{ms} v^m. \qquad \ldots(4.12)$$

we can derive

$$M_{s|k} = (\gamma_k + \phi_k) M_s. \qquad \ldots(4.13)$$

By virtue of eqn. (4.9), eqn. (4.13) may be replaced by

$$M_{s|k} = \beta M_k M_s. \qquad \ldots(4.14)$$

Hence, in view of (4.1) on (4.10), we can find

$$M_{s|k} v^k = 0. \qquad \ldots(4.15)$$

At this moment, using the latter part of (4.12) and (2.7), we get

$$M_s v^s (\phi_m v^m)_{|s} v^s. \qquad \ldots(4.16)$$

In view of the identity $(\phi_m v^m)_{|s} v^s = 0$, we can obtain (4.10).

On the other hand, under an affine motion, we have always,

$$\mathcal{L}_v \gamma_s = \gamma_{s|m} v^m + \gamma_m v^m_{|s} = 0 \qquad \ldots(4.17)$$

or

$$\mathcal{L}_v \gamma_s = (\gamma_{s|m} + \gamma_m \phi_s) v^m = 0. \qquad \ldots(4.18)$$

On substituting the value of γ_s from (4.9) into the last result, we have

$$\begin{aligned}(\beta M_s - \phi_s)_{|m} v^m &+ (\beta M_m - \phi_m) \phi_s v^m \\ &= \beta_{|m} M_s v^m + \beta M_{s|m} v^m - \phi_{s|m} v^m + \beta M_m \phi_s v^m - \phi_m \phi_s v^m = 0 \\ &= M \mathcal{L}_v \beta(x) + \beta M_{s|m} v^m - (\phi_{s|m} v^m + \phi_m \phi_s v^m) \\ &\qquad + \beta M_m \phi_s v^m = 0. \qquad \ldots(4.19)\end{aligned}$$

Hereupon, if we take the case of (3.3), (4.10) and (4.15), we obtain the remarkable property :

$$M_s \mathcal{L}v\beta(x) = 0. \qquad \ldots(4.20)$$

From (4.20), in case of (4.1), we have

$$\mathcal{L}v\beta(x) = 0^{(3)}. \qquad \ldots(4.21)$$

5. Recurrent Case

The author has also studied the recurrent affine motion in an F_n^*-space (Kumar 1976c). The basic condition is

$$\phi_h + \gamma_h = 0. \qquad \ldots(5.1)$$

This gives

$$\phi_h + \gamma_h = 0 \rightarrow K_{hs}v^s = 0 \rightarrow K_{hs} = \gamma_h \epsilon_s \text{ or } \gamma_h v^h = 0 \qquad \ldots(5.2)$$

where ϵ_s means a suitable covariant gradient vector defined by

$$\epsilon_s \equiv \frac{1}{\epsilon} \partial_s \epsilon, \ \epsilon = \epsilon(x) \qquad \ldots(5.3)$$

and satisfies the following relations :

(a) $\epsilon_s v^s = 0$

and

(b) $\epsilon_{h|s} = \epsilon_h \epsilon_s$. ...(5.4)

Introducing the vector M_s we get

$$M_s = (\phi_h v^h)_{|s} = \sigma_{|s}, \qquad \ldots(5.5)$$

where

$$\sigma \equiv \phi_h v^h. \qquad \ldots(5.6)$$

Hence, M_s denotes a gradient vector.

By virtue of (4.10), (4.14), (5.5) and the gradience property of ϵ_h and M_h, we can conclude that M_h is a vector similar to ϵ_h.

Let us remember the integrability condition for an affine motion

$$\mathcal{L}vK^i_{hjk} = -\phi_s v^s K^i_{hjk} - \phi_s v^i K^s_{hjk} + \phi_h v^s K^i_{sjk} + \phi_j v^s K^i_{hsk} + \phi_k v^s K^i_{hjs} = 0. \qquad \ldots(5.7)$$

(3) This will be shown by straight Lie-derivation of (4.9) :

$$M_h \mathcal{L}v\beta = \mathcal{L}v\phi_h + \mathcal{L}v\gamma_h = 0, \text{ say } \mathcal{L}v\beta(x) = 0.$$

Contracting eqn. (5.7) with respect to indices i and k, we have

$$-\phi_s v^s K_{hj} + \phi_h v^s K_{sj} = 0 \qquad ...(5.8)$$

where we have used (1.6), (2.1b) and (4.1). In veiw of (4.12), eqn. (5.8) may be written as

$$\phi_s v^s K_{hj} = \phi_h M_j. \qquad ...(5.9)$$

Now, in the present case, $\phi_s v^s$ denotes a non-constant scalar function $\sigma(x)$. Then, we may assume the resolvability of K_{hj} of the form

$$K_{hj} = \gamma_h M_j,\ \gamma_h = \frac{1}{\sigma}\ \phi_h. \qquad ...(5.10)$$

On the other hand, introducing the form of γ_h from (5.10) into (4.1), we get

$$\left(\frac{1}{\sigma}\phi_h + \phi_h\right)v^h = 0. \qquad ...(5.11)$$

or

$$(1 + \sigma)\phi_h v^h = 0. \qquad ...(5.12)$$

Since $(1 + \sigma) \neq 0$, we have $\phi_h v^h = 0$; this contradicts our assumption $\phi_h v^h = \sigma \neq 0$.

In the second step, we try to resolve K_{hj} in the form :

$$K_{hj} = \gamma_h \xi_j \qquad ...(5.13)$$

where

$$\text{(a)} \qquad \xi_j \equiv -\frac{1}{\sigma} M_j$$

and

$$\text{(b)} \qquad \xi_{j\,|\,s} = \xi_j \xi_s. \qquad ...(5.14)$$

With the help of eqns. (5.9) and (5.12), comparing the two forms of $K_{hj}(x, \dot{x})$, we obtain

$$K_{hj} = \frac{1}{\sigma}\ \phi_h M_j = \gamma_h\left(-\frac{1}{\sigma}\ M_j\right). \qquad ...(5.15)$$

or

$$\frac{1}{\sigma}\ (\phi_h + \gamma_h)M_j = 0. \qquad ...(5.16)$$

Here, we are trying to have $(\phi_h + \gamma_h) = 0$ for a non-zero M_j, i.e., we want to get a set which we have shown in the relation

$$v^i_{\ |\,j} = -\gamma_j v^i,\ K_{hj} = \gamma_h \xi_j,\ \xi_{j\,|\,s} = \xi_j \xi_s. \qquad ...(5.17)$$

If this is possible because of (4.10), we have

$$\xi_h v^h = 0. \qquad \ldots(5.18)$$

Such a case has been considered in another paper (Kumar 1976b).

We shall assume this fact and seek a necessary condition for this case implied in our theory.

Equation (5.14*a*) can also be written as

$$M_j = -\sigma\xi_j. \qquad \ldots(5.19)$$

Differentiating eqn. (5.19) covariantly with respect to x^s and noting eqns. (5.4), (5.14*b*) and (5.19), we get

$$M_{j|s} = -M_s\xi_j - \sigma\xi_j\xi_s = -M_s\xi_j - (\sigma\xi_s)\,\xi_j = -M_s\xi_j + M_s\xi_j = 0. \qquad \ldots(5.20)$$

On one hand, from (4.14), we can say that in order that the present theory contains the main theory of the author's earlier paper (Kumar 1976c), it is necessary that we have $\beta = 0$. Further, it satisfies (4.21). Hence, it is reasonable to consider such a case.

Conversely, if $\beta = 0$, we get $(\phi_h + \gamma_h) = 0$.

Thus, we conclude as follows.

Theorem 5.1—In order that a recurrent affine motion, in an F_n^*-space

$$\bar{x}^i = x^i + v^i(x)\,dt,\ v^i_{\ |j} = \phi_j v^i$$

admitted under assumptions

$$K^s_{hjs}\,v^j = 0 \text{ and } (\gamma_h + \phi_h)v^h = 0$$

becomes a main motion of the same kind

$$\bar{x}^i = x^i + v^i(x)\,dt,\ v^i_{\ |j} = -\gamma_j v^i,\ K_{hj} = \gamma_h\xi_j,\ \xi_{j|s} = \xi_j\xi_s$$

it is necessary and sufficient that we have $\beta(x) = 0$, where $\beta(x)$ denotes a proportional factor, such that

$$\gamma_h + \phi_h = \beta(x)K_{sh}v^s \text{ and } \mathcal{L}_v\beta(x) = 0.$$

6. Appendices to Section 3

In the case of $\phi_j =$ non-gradient vector, we can state the following.

Theorem 6.1—When $\phi_h v^h$ = constant and ϕ_j is not a gradient vector, the space F_n^* is able to admit a recurrent motion of the form

$$\bar{x}^i = x^i + v^i(x)\,dt,\ v^i_{|j} = \phi_j v^i \ \text{ with } \gamma_h v^h = 0.$$

The process of calculation developed in sections 1 and 2 holds when ϕ_j is replaced by $-\gamma_j$ formally. Hence, we have the Theorem (6.1).

In the existing theory of special recurrent affine motion of the form

$$\bar{x}^i = x^i + v^i(x)\,dt,\ v^i_{|j} + \gamma_j v^i = 0$$

if we put an additional condition

$$\gamma_h v^h = \text{constant},$$

we can obtain the following two independent categories of the motion

$$\bar{x}^i = x^i + v^i(x)\,dt,\ v^i_{|j} + \gamma_j v^i = 0 \quad \text{...(6.1)}$$

satisfying

$$\gamma_h v^h \neq 0,\ \gamma_{h|s} = \gamma_{s|h} \text{ and } K_{hj} = 0$$

or

$$\bar{x}^i = x^i + v^i(x)\,dt,\ v^i_{|j} + \gamma_j v^i = 0 \quad \text{...(6.2)}$$

satisfying

$$\gamma_h v^h = 0. \quad \text{...(6.3)}$$

These two results have been reported elsewhere (Kumar 1976c). The former occurs when $\epsilon_s = 0$.

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REFERENCES

Eisenhart, L. P. (1927). Non-Riemannian Geometry. New York.

Kumar, A. (1976a). Some theorems on affine motion in a recurrent Finsler space. (Communicated).

——— (1976b). Some theorems on affine motion in a recurrent Finsler space II. (Communicated).

——— (1976c). Some theorems on affine motion in a recurrent Finsler space III. (Communicated).

Rund, H. (1959). The Differential Geometry of Finsler Space. Springer Verlag, Berlin.

Takano, K. (1961). Affine motion in non-Riemannian K^*-space IV. *Tensor, N.S.*, **11**, 245-53.

Yano, K. (1957). The Theory of Lie-derivatives and Its Application. Amsterdam.