BINORMAL OPERATORS

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The object of this paper is to prove the following results for the operators of class $B(N)$.

Theorem 1—Let $T \in B(N)$ and $S \in B(N)$. If T and S are doubly commuting then TS is binormal.

Theorem 2—$B(N)$ does not contain all subnormal operators.

Theorem 3—There exists a K-paranormal binormal operator which is not hyponormal.

Theorem 4—Let T be an idempotent operator and let $T \in B(N)$. Then T is self-adjoint.

By an operator in this paper, we shall mean a bounded linear transformation of a Hilbert space H into itself. An operator T is said to be binormal if T^*T commutes with TT^* (Stephen 1972). Let $B(N)$ denote the set of all binormal operators. The object of this paper is to prove some results for the operators of class $B(N)$.

An operator T is said to be quasi-normal if $TT^*T = T^*T^2$; subnormal if T has a normal extension, hyponormal if $TT^* \leqslant T^*T$, paranormal if $\| Tx \|^2 \leqslant \| T^2x \|$ for all unit vectors x in H. According to Istratescu (1967) T is said to be k-paranormal $(k \geqslant 2)$ if $\| Tx \|^k \leqslant \| T^k x \|$ for all unit vectors x in H. An operator T is said to be normaloid if $\| T \|^n = \| T^n \|$, convexoid if $\overline{W(T)} = \text{con. } \sigma(T)$, spectraloid if $r(T) = w(T)$ where $\overline{W(T)}$, con. $\sigma(T)$, $w(T)$ and $r(T)$, denote the closure of the numerical range $W(T)$ of T, convex hull of the spectrum $\sigma(T)$ of T, numerical radius of T, and spectral radius of T respectively (Halmos 1967). Let $N(T) = \{x : Tx = 0\}$ be the null space of T.

Campbell (1975) has proved that if $T \in B(N)$ then T^2 need not belong to $B(N)$. Thus the product of two commuting binormal operators need not be binormal. However if they are doubly commuting then we have;

Theorem 1—Let $T \in B(N)$ and $S \in B(N)$. If T and S are doubly commuting then TS is binormal.

PROOF: $(TS)\,(TS)^*(TS)^*(TS)$

$$\begin{aligned}
&= TS\ S^*T^*T^*S^*TS \\
&= ST\ S^*T^*T^*TS^*S \\
&= SS^*\ TT^*T^*TS^*S \\
&= SS^*T^*TTT^*S^*S \qquad \text{(since } T \in B(N)\text{)} \\
&= ST^*S^*TTS^*T^*S \\
&= T^*STS^*S^*TST^* \\
&= T^*TSS^*S^*STT^* \\
&= T^*TS^*SSS^*TT^* \qquad \text{(since } S \in B(N)\text{)} \\
&= T^*S^*TSSTS^*T^* \\
&= (TS)^*(TS)\,(TS)\,(TS)^*
\end{aligned}$$

Hence TS is binormal.

We know that $B(N)$ contains quasi-normal operators. This raises the following question:

Does $B(N)$ contain all subnormal operators?

The answer is in negative as can be seen by the following:

Theorem 2*—$B(N)$ does not contain all subnormal operators.

PROOF: Suppose $B(N)$ contains all subnormal operators. Consider a unilateral shift operator U. We know that U is binormal and also U is subnormal (Halmos 1967). However U being subnormal, $U+\lambda$ is also subnormal for any scalar λ, and hence $U+\lambda \in B(N)$ by our assumption. Since U and $U+\lambda$ are both binormal, it follows from Campbell (1975) that U is normal, a contradiction.

Campbell (1975) has proved the following theorem: If T is binormal, then T is hyponormal if it is paranormal. That this result cannot be generalised to k-paranormal operators, is shown by the following:

Theorem 3—There exists a k-paranormal binormal operator $k > 2$ which is not hyponormal.

PROOF: Let T be the bilateral weighted shift operator with weights $\{\alpha_n\}$, where

$$\alpha_n = \tfrac{1}{2} \text{ if } n \leqslant -1$$

$$= \frac{1}{\sqrt{3}} \text{ (if } n = 0) = \frac{n}{n+1} \text{ if } n \geqslant 1.$$

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* Theorem 2 can also be proved as follows. If $T \in B(N)$ is subnormal and non-normal, then $T + I$ is subnormal (Halmos 1967) but $T + I \notin B(N)$ (Campbell 1972).

Clearly T is binormal. Patel (1975) has shown that a bilateral weighted shift operator T is k-paranormal if and only if $|\alpha_n|^{k-1} \leqslant |\alpha_{n+1}| \cdot |\alpha_{n+2}| \ldots |\alpha_{n+k-1}|$. Thus T is k-paranormal. As $\alpha_0 > \alpha_1$, T cannot be hyponormal (Halmos 1967).

We observe that if $T^2 = 0$ then $T \in B(N)$ trivially. But if $T^n = 0$ for $n > 2$ then T need be binormal. For example take

$$T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $T^3 = 0$ but $T \notin B(N)$.

Also a binormal operator may not be convexoid. For example

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Furuta and Nakamoto (1971) have proved that an idempotent operator is a projection if it is normaloid or spectraloid. Since the class $B(N)$ is independent of the class of normaloid and of the class of spectraloid operators, it is interesting to know whether an idempotent operator which is binormal is a projection. Here we show that the answer is in affirmative.

Theorem 4—Let T be an idempotent operator and let $T \in B(N)$. Then T is self-adjoint.

PROOF: We first show that $N(T)$ reduces T. Since $N(T)$ is invariant under T, therefore, it will suffice to prove that $N(T)$ is invariant under T^*.

Since $T^2 = T$ and $T \in B(N)$

therefore $T^*TT^* = TT^*T$.

Hence for $x \in N(T)$, $T^*TT^*x = 0$.

Then $(TT^*)^2 x = 0$.

Since TT^* is positive,

$TT^*x = 0$

which implies that $T^*x \in N(T)$.

Thus $N(T)$ is invariant under T^*.

Now we complete the proof by asserting that $T^* = T^*T$.

Since $T^2 = T$, therefore $T(I - T)x = 0$ for $x \in H$.

This shows that $(I - T)x \in N(T)$, and hence

$T^*(I - T)x \in N(T)$ as $N(T)$ reduces T.

Therefore $TT^*(I - T)x = 0$.

Thus

$$T^*(I - T)x = 0 \text{ as } N(TT^*) = N(T^*).$$

Hence $T^* = T^*T$.

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References

Campbell, Stephen L. (1972). Linear operators for which T^*T and TT^* commute. *Proc. Am. math. Soc.*, **34**, 177–80.

———— (1975). Linear operators for which T^*T and TT^* commute II. *Pac. J. Math.*, (to appear).

Furuta, T., and Nakamoto, Ritsuo (1971). Certain numerical radius contraction operators. *Proc. Am. math. Soc.*, **29**, 521–24.

Halmos, P. R. (1967). A Hilbert Space Problem Book, Princeton University Press.

Istratescu, V. (1967). On some classes of operators. *Proc. Japan Acad.*, **43**, 603–606.

Patel, S. M. (1975). On class of non-hyponormal operators. *Math. Natr.* (to appear).