PARACOMPACTNESS IN BITOPOLOGICAL SPACES AND AN APPLICATION TO QUASI-METRIC SPACES

by M. C. DATTA, *Department of Mathematics, Birla Institute of Technology and Science, Pilani*

(*Received* 9 *October* 1974; *after revision* 4 *November* 1976)

In this paper we give a definition of 'paracompactness' for bitopological spaces. The classical theorem of Dieudonné, that is, a paracompact space is normal is generalized to bitopological spaces. In order to obtain equivalences (for paracompactness in bitopological spaces) analogous to the conventional ones (cf. Michael) for general topological spaces we obtain a theorem which has a very pleasant consequence to topological spaces. An unsolved question of Stoltenberg, namely "When is a quasi-metric topological space paracompact?" is being answered (partially) by means of this theorem.

1. INTRODUCTION

The theory of bitopological spaces started with the paper of Kelly (1963). Separation axioms in bitopological spaces were first studied by him. In general, the pattern of these axioms has been to mix the two topologies of the bitopological space in a certain way. For example, a space (X, τ_1, τ_2) is said to be pairwise Hausdorff, if two distinct points can be separated by disjoint τ_1-open and τ_2-open sets; pairwise regular, if a τ_1-closed (τ_2-closed) set and a point not belonging to this set can be separated by disjoint τ_2-open (τ_1-open) and τ_1-open (τ_2-open) sets; and finally pairwise normal, if disjoint τ_1-closed and τ_2-closed sets can be separated by disjoint τ_2-open and τ_1-open sets respectively.

In the same style, the definition of pairwise paracompactness was attempted by Fletcher *et al.* (1969) as follows. A bitopological space (X, τ_1, τ_2) is pairwise paracompact, provided it is pairwise Hausdorff, every τ_1-open cover has a τ_2-open, τ_2-locally finite refinement, and every τ_2-open cover has a τ_1-open, τ_1-locally finite refinement. But in this case, it turns out that $\tau_1 = \tau_2$ and the resulting single topology is paracompact, whenever (X, τ_1, τ_2) is pairwise paracompact.

In this paper, we attempt a different definition of pairwise paracompactness and avoid the pitfall into which the above definition has fallen. With our definition, we are able to generalize the classical theorem of Dieudonné, i.e., a paracompact space is normal. In our attempt to continue this theory and obtain equivalences (for paracompactness in bitopological spaces) analogous to the conventional ones (cf. Michael 1953) for general topological spaces, we obtain a theorem [cf. Theorem (2.10)] which has a very pleasant consequence. An unsolved question of Stoltenberg (1969), namely, "When is a quasi-metric topological space paracompact?" is capable of being answered (partially) by means of our Theorem (2.10).

2. Pairwise Paracompact Spaces

Fletcher *et al.* (1969) defined a cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2) as pairwise open if $\mathcal{U} \subset \tau_1 \cup \tau_2$, $\tau_1 \cap \mathcal{U}$ contains a non-empty set and $\tau_2 \cap \mathcal{U}$ contains a non-empty set. Now, we define a weaker concept.

Definition 2.1 — A cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2) is said to be weakly pairwise open if it consists of either τ_1-open sets or τ_2-open sets or both.

Pairwise closed and weakly pairwise closed covers are defined analogously.

Let $\mathcal{U} = (U_\alpha)_{\alpha \in \Lambda}$ be a (weakly) pairwise open cover of a bitopological space (X, τ_1, τ_2). For any such cover, the following shall always be stipulated : A member U_β of $\mathcal{U}$ which is both τ_1-open and τ_2-open shall once for all be designated as either a τ_1-open set only or a τ_2-open set only. This means that though a certain U_β may be both τ_1-open and τ_2-open, we shall consider it either as a τ_1-open set or a τ_2-open set throughout. When one of these alternatives is unambiguously chosen for each member of $\mathcal{U}$, which is both τ_1-open and τ_2-open (the alternatives may differ from member to member), we shall say that the (weakly) pairwise open cover $\mathcal{U}$ is well defined.

Throughout what follows, all (weakly) pairwise open covers considered will be assumed to be well defined (without explicit mention), and any construction of (weakly) pairwise open covers will be deemed to have been verified to give rise to a well defined one.

Definition 2.2 — Let (X, τ_1, τ_2) be a bitopological space. Let $\mathcal{U} = (U_\alpha)_{\alpha \in \Lambda}$ be a (weakly) pairwise open cover of X. Then the (weakly) pairwise open cover $\mathcal{V} = (V_\beta)_{\beta \in \Gamma}$ is said to be a parallel refinement of $\mathcal{U}$ if every τ_1-open set V_β is contained in some τ_1-open set U_α and every τ_2-open set V_β is contained in some τ_2-open set U_α.

Definition 2.3 — Let $\mathcal{U} = (U_\alpha)_{\alpha \in \Lambda}$ be a (weakly) pairwise open cover of a bitopological space (X, τ_1, τ_2). Then the (weakly) pairwise closed covering $\mathcal{F} = (F_\beta)_{\beta \in \Gamma}$ is said to be a parallel refinement of $\mathcal{U}$ if every τ_1-closed set F_β is contained in some τ_2-open set U_α and every τ_2-closed set F_β is contained in some τ_1-open set U_α.

Definition 2.4 — A pairwise open cover $\mathcal{U}$ or a weakly pairwise open cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2) is said to be pairwise locally finite (p.l.f.) if for every $x \in X$, there exists a τ_1-open (τ_2-open) neighbourhood of x which meets only a finite number of τ_2-open (τ_1-open) sets of $\mathcal{U}$.

Definition 2.5 — Given a pairwise open cover $\mathcal{U}$ or a weakly pairwise open cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2), a refinement $\mathcal{V}$ of $\mathcal{U}$ is said to be p.l.f. if for every $x \in X$ there exists a τ_1-open (τ_2-open) neighbourhood of x, which meets only a finite number of sets of $\mathcal{V}$, which form a refinement of the class of τ_2-open (τ_1-open) sets of $\mathcal{U}$.

Definition 2.6 — A bitopological space (X, τ_1, τ_2) is said to be (strongly) pairwise paracompact if

(i) (X, τ_1, τ_2) is pairwise Hausdorff; and

(ii) for every (weakly) pairwise open cover of X there exists a p.l.f. (weakly) pairwise open parallel refinement.

Remark 1 : Every pairwise Hausdorff, pairwise compact space is pairwise paracompact. Every strongly pairwise paracompact space is pairwise paracompact.

Proposition 2.7 — Let (X, τ_1, τ_2) be a pairwise paracompact space and let F be a τ_1-closed subset of X. Then,

(a) F is pairwise paracompact; and

(b) Given any τ_2-open cover $\mathcal{U}$ of F there exists a τ_1 locally finite τ_2-open refinement of $\mathcal{U}$.

A similar result holds for a τ_2-closed set.

The proof follows in a standard way, as in the classical case, that a closed subset of a paracompact space is paracompact.

Now we shall prove a generalization of Dieudonné's theorem that every paracompact space is normal. For the proof of this, we need the following :

Lemma 2.8 — Let (X, τ_1, τ_2) be a pairwise paracompact space. Let A and B be two disjoint subsets of X such that A is τ_1-closed and B is τ_2-closed. If for each $x \in A$ there is a τ_2-open neighbourhood V_x of x, a τ_1-open neighbourhood W_x of B which do not intersect, then there exist a τ_1-open neighbourhood T of A and a τ_1-open neighbourhood U of B which do not intersect.

PROOF : Consider the pairwise open cover of X consisting of $(X - A)$ and the V_x, where $x \in A$. Since (X, τ_1, τ_2) is pairwise paracompact, there exists a p.l.f. pairwise open parallel refinement $(T_i)_{i \in I}$ of this covering. If $A \cap T_i \neq \phi$, there exists $x_i \in A$, such that $T_i \subset V_{x_i}$. Let T be the τ_2-open set which is the union of T_i's which meet A. We shall show that there is a τ_1-open neighbourhood U of B which does not meet T. For every $y \in B$ there is a τ_1-open neighbourhood S_y of y which meets only a finite number of sets of $(T_i)_{i \in I}$ which meets A. Let J be the finite subset of I consisting of these indices i, such that T_i meets both S_y and A. If we put

$U_y = S_y \cap W_{x_i}$, then U_y is a τ_1-open neighbourhood of y which meets none of the T_i's which meet A and hence $U_y \cap T = \phi$. Let $U = \bigcup_{y \in B} U_y$. Then, U is a τ_1-open neighbourhood of B which does not meet T.

Theorem 2.9 — Every pairwise paracompact space is pairwise normal.

The proof follows from Lemma 2.8.

Theorem 2.10 — For a pairwise regular space (X, τ_1, τ_2) consider the following statements.

(a) X is pairwise paracompact.

(b) Every pairwise open cover of X has a p.l.f. refinement [p.l.f. in the sense of Definition (2.5)].

(c) Every pairwise open cover of X has a pairwise closed p.l.f. refinement, which is parallel in the sense of Definition (2.3).

(a′) X is strongly pairwise paracompact.

(b′) Every weakly pairwise open cover of X has a p.l.f. refinement [p.l.f. in the sense of Definition (2.5)].

(c′) Every weakly pairwise open cover of X has a weakly pairwise closed p.l.f. refinement, which is parallel in the sense of Definition (2.3).

(d) (X, τ_1, τ_2) is τ_1-paracompact and τ_2-paracompact.

Then $(a') \Rightarrow (a) \Rightarrow (b) \Rightarrow (c)$ and $(a') \Rightarrow (b') \Rightarrow (c')$. Further, if (X, τ_1) and (X, τ_2) are both Hausdorff, then $(c') \Rightarrow (d)$; if (X, τ_1) is Hausdorff then $(c') \Rightarrow$ (X, τ_1, τ_2) is τ_1-paracompact.

PROOF: $(a') \Rightarrow (a)$. This follows from Definition (2.6).

$(a) \Rightarrow (b)$. This is trivial.

$(b) \Rightarrow (c)$. Let $\mathcal{R}$ be any pairwise open cover of X. For each $x \in X$ there is a τ_1-open set $U \in \mathcal{R}$ or a τ_2-open set $V \in \mathcal{R}$, which contains x and therefore (since X is pairwise regular) we have

(i) If $x \in U$, there exists a τ_1-open set W_x such that

$$x \in W_x \subset \overline{W}_x^{\tau_2} \subset U, \text{ or } (*)$$

(ii) If $x \in V$, there exists a τ_2-open set Z_x such that

$$x \in Z_x \subset \overline{Z}_x^{\tau_1} \subset V.$$

The family $\mathcal{B}$ consisting of $(W_x \text{ or } Z_x)_{x \in X}$ is a pairwise open covering of X. By hypothesis, there exists a p.l.f. covering $\mathcal{D}$ of X which is finer than $\mathcal{B}$. Let $\mathcal{F}$ be the

(*) $\overline{W}_x^{\tau_2}$ denotes τ_2-closure of W_x and $\overline{Z}_x^{\tau_1}$ denotes the τ_1-closure of Z_x.

family obtained as follows: If $D \in \mathcal{D}$ is contained in some $Z_x \in \mathcal{B}$ take the τ_1-closure of D and if $D \in \mathcal{D}$ is contained in some $W_x \in \mathcal{B}$, take the τ_2-closure of D. $\mathcal{F}$ is the family of all these closures. Since the family $\mathcal{B}'$ consisting of $(W_x^{\tau_2} \text{ or } Z_x^{\tau_1})_{x \in X}$ is finer than $\mathcal{R}$, it follows that $\mathcal{F}$ is a pairwise closed refinement of $\mathcal{R}$. We shall show that $\mathcal{F}$ is p.l.f. Given $x \in X$, there exists a τ_2-open neighbourhood Z of x such that Z meets only a finite number of sets of $\mathcal{D}$ which refine the class of τ_1-open sets of $\mathcal{B}$. If a τ_2-open neighbourhood Z does not meet some D it also does not meet D^{τ_2}. Again, for every $x \in X$ there exists a τ_1-open neighbourhood which meets only a finite number of sets of $\mathcal{D}$ which refine the class of τ_2-open sets of $\mathcal{B}$. If a τ_1-open neighbourhood Z does not meet some D then it also does not meet D^{τ_1}. Hence $\mathcal{F}$ is p.l.f. and it is clear that it preserves the 'parallel' property, as required.

$(a') \Rightarrow (b')$. This is trivial.

$(b') \Rightarrow (c')$. If in the proof for $(b) \Rightarrow (c)$ we replace 'pairwise open' by 'weakly pairwise open' and 'pairwise closed' by 'weakly pairwise closed' we get the proof of $(b') \Rightarrow (c')$. $(c') \Rightarrow (d)$. Let $\mathcal{R}$ be any τ_1-open cover of X. Let $\mathcal{U}$ be a p.l.f. (i.e. in this case τ_2-locally finite) cover which refines $\mathcal{R}$. For each $x \in X$, let Z_x be a τ_2-open neighbourhood of x which meets only a finite number of sets of $\mathcal{U}$. The family $\mathcal{B} = (Z_x)_{x \in X}$ is a τ_2-open cover of X. Let $\mathcal{F}$ be a p.l.f. weakly pairwise closed covering of X which refines $\mathcal{B}$ in a 'parallel' way as per Definition (2.3), i.e., $\mathcal{F}$ is τ_1-locally finite τ_1-closed cover of X.

For each $A \in \mathcal{U}$, let U_A be a set of $\mathcal{R}$ which contains A. Let D_A be the union of the τ_1-closed set $F \in \mathcal{F}$ such that $A \cap F = \phi$. Since $\mathcal{F}$ is τ_1-locally finite, D_A is τ_1-closed in X. Let $A_1 = U_A \cap (X - D_A)$.

Since we have $A \cap D_A = \phi$ and $A \subset U_A$, it follows that $A \subset A_1$ and, therefore, the family $\mathcal{A}$ of sets A_1 as A runs through $\mathcal{U}$ is a τ_1-open cover of X. Moreover, since $A_1 \subset U_A \in \mathcal{R}$, $\mathcal{A}$ is a refinement of $\mathcal{R}$. It remains to be shown that $\mathcal{A}$ is τ_1-locally finite.

For each $x \in X$ there is τ_1-open neighbourhood T of x which meets only a finite number of τ_1-closed sets of $\mathcal{F}$, because $\mathcal{F}$ is τ_1-locally finite. Let these finite number of sets of $\mathcal{F}$ be $F_1, F_2, \ldots, F_n$. Since each F_i is contained in a set of the form Z_{x_i}, and Z_{x_i} meets a finite number of sets of $\mathcal{U}$, it follows that each F_i meets a finite number of sets of $\mathcal{U}$. Call these sets A_{ij} $(1 \leqslant j \leqslant s_i)$. If A is a set of $\mathcal{U}$ other than one of the A_{ij} $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant s_i)$, it follows that A_1 meets none of the F_i, and, therefore, does not meet $T \subset \bigcup_{i=1}^{n} F_i$. This proves that X is τ_1-paracompact.

Similarly, X is τ_2-paracompact.

Remark 2 : The implication $(a') \Rightarrow (d)$ turns out to be useful in answering the question posed by Stoltenberg (1969) : When is a quasi-metric space paracompact? If (X, τ_1) is the quasi-metric space and τ_2 is the topology conjugate to τ_1 determined by the conjugate quasi-metric, then the bitopological space (X, τ_1, τ_2) is pairwise regular and so the implication $(a') \Rightarrow (d)$ of Theorem (2.10) is applicable. This means that if (X, τ_1, τ_2) is strongly pairwise paracompact and (X, τ_1) is Hausdorff then (X, τ_1) is paracompact. So we get the following :

Theorem 2.11 — A sufficient condition that a quasi-metric space be paracompact is that it should be Hausdorff and the corresponding bitopological space is strongly pairwise paracompact.

Remark 3 : Though the statement of this theorem seems to imply that from a 'strong' property we infer a 'weak' property, in reality it is not so. The non-triviality of the result can be seen from the fact that the hypothesis gives among other things, a τ_1-open τ_2-locally finite refinement from a τ_1-open cover, whereas the conclusion gives a τ_1-open τ_1-locally finite refinement from a τ_1-open cover.

Remark 4 : However, the above condition is not necessary. For, consider the space (R, τ_1, τ_2), where R is the set of real numbers and τ_1 and τ_2 are topologies on R whose basic open sets are of the form $[a, b)$ and $(c, d]$ respectively. (R, τ_1, τ_2) is τ_1-paracompact and τ_2-paracompact. But it is not strongly pairwise paracompact, because consider the covering $[0, \frac{1}{2})$, $[\frac{1}{2}, \frac{3}{4})$, ..., $\left[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n}\right)$, ... and $[1, 2)$ and the remaining sets of the covering consisting of any τ_1-open and τ_2-open sets. Take any parallel pairwise open refinement of this covering. It is not p.l.f., because every τ_2-open neighbourhood of 1 meets an infinite number of any such refinement. Therefore, it is not strongly pairwise paracompact.

Remark 5 : Though the above application of the theory of bitopological spaces to general topological spaces is very interesting, one wishes, after going through the complexity of Theorem (2.10), that a suitable definition of paracompactness in bitopological spaces may be found which will lead to all the equivalences of Michael (1953) and also to an analogue of the theorem of A. H. Stone that every metric space is paracompact.

Acknowledgement

The author expresses his deep sense of gratitude to Prof. V. Krishnamurthy for his encouragement and help in the preparation of this paper.

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