DEGREE OF APPROXIMATION OF FUNCTIONS BY MODIFIED BERNSTEIN POLYNOMIAL ON AN UNBOUNDED INTERVAL

by ANWAR HABIB *and* ABDUL WAFI, *Department of Mathematics, Aligarh Muslim University, Aligarh*

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Chlodovsky (1937) has proved the Theorems 1.1 and 1.2 for the Bernstein polynomials

$$B_n(x) = B_n^f(x; b_n) = \sum_{k=0}^{n} f\left(\frac{b_n k}{n}\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1 - \frac{x}{b_n}\right)^{n-k}$$

on an unbounded interval.

The object of this paper is to extend the above theorems for modified Bernstein polynomials

$$P_n(x) = P_n^f(x; b_n)$$

$$= (n+1)\sum_{k=0}^{n}\left(\int_{k/(n+1)}^{(k+1)/(n+1)} f(tb_n)\,dt\right) p_{n,k}\left(\frac{x}{b_n}\right),$$

on an unbounded interval.

1. Introduction and Results

If $f(x)$ is a function defined on [0, 1], the Bernstein polynomial $B_n^f(x)$ of f is

$$B_n^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k(1 - x)^{n-k}.$$

If the function $f(x)$ be defined on the interval $(0, b)$, $b > 0$, the Bernstein polynomial $B_n^f(x; b)$ for this interval is given by

$$B_n(x) = B_n^f(x; b) = \sum_{k=0}^{n} f\left(\frac{bk}{n}\right)\binom{n}{k}\left(\frac{x}{b}\right)^k\left(1 - \frac{x}{b}\right)^{n-k}.$$

A small modification of Bernstein polynomial $B_n^f(x)$ due to Kantorovič (1930) makes it possible to approximate Lebesque integrable function in the L_1-norm by the modified polynomial

$$P_n^f(x) = (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right) p_{n,k}(x).$$

Let the function $f(x)$ be defined on the interval $(0, b)$, $b > 0$. To obtain the modified Bernstein polynomial $P_n^f(x; b)$ for this interval, we make the substitution $y = xb^{-1}$ in the polynomial $P_n^\phi(y)$ of the function $\phi(y) = f(by)$, $0 \leqslant y \leqslant 1$ and obtain in this way

$$\begin{aligned} P_n(x) &= P_n^f(x; b) \\ &= (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} f(tb)\, dt \right) p_{n,k}\left(\frac{x}{b}\right), \end{aligned}$$

where

$$p_{n,k}\left(\frac{x}{b}\right) = \binom{n}{k} \left(\frac{x}{b}\right)^k \left(1 - \frac{x}{b}\right)^{n-k}.$$

Chlodovsky (1937) has proved the following theorems by assuming that $b = b_n$ is a function of n, which increases to $+\infty$ with n, and $f(x)$ is defined in the infinite interval $0 \leqslant x < +\infty$.

Theorem 1.1 — If $b_n = o(n)$ and the function $f(x)$ is bounded in $(0, +\infty)$, say $|f(x)| \leqslant M$, then $B_n(x) \to f(x)$ holds at any point of continuity of the function f.

Theorem 1.2 — If $b_n = o(n)$

and

$$M(b_n)\, e^{-\alpha n/b_n} \to 0,$$

for each $\alpha > 0$, then $B_n(x) \to f(x)$ holds at each point of continuity of the function $f(x)$.

In this paper, our object is to improve the above results by taking the modified polynomial $P_n^f(x; b)$ instead of $B_n^f(x; b)$, which may be stated as follows :

Theorem 1.3 — If $b_n = o(n)$ and the function $f(x)$ is bounded Lebesque integrable in $(0, +\infty)$, say $|f(x)| \leqslant M$, then $P_n(x) \to f(x)$ holds at any point of continuity of the function f.

Theorem 1.4 — If $b_n = o(n)$

and

$$M(b_n)\, e^{-\alpha n/b_n} \to 0, \qquad \ldots(1.1)$$

for each $\alpha > 0$, then $P_n(x) \to f(x)$ holds at each point of continuity of the integrable function $f(x)$.

2. Proofs of the Theorems

Proof of Theorem 1.3 : We have

$$| P_n(x) - f(x) |$$

$$\leqslant (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} | f(b_n t) - f(x) | \, dt \right) p_{n,k}\left(\frac{x}{b_n}\right).$$

Let $\epsilon > 0$ be arbitrary and choose $\delta > 0$ so small that $|f(x) - f(x')| < \epsilon$ for $|x - x'| < \delta$, then we have

$$| P_n(x) - f(x) |$$

$$\leqslant (n+1) \sum_{|b_n t - x| < \delta} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} | f(b_n t) - f(x) | \, dt \right) p_{n,k}\left(\frac{x}{b_n}\right)$$

$$+ (n+1) \sum_{|b_n t - x| \geqslant \delta} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} | f(b_n t) - f(x) | \, dt \right) p_{n,k}\left(\frac{x}{b_n}\right)$$

$$= I_1 + I_2. \qquad \ldots(2.1)$$

$$I_1 = (n+1) \sum_{|b_n t - x| < \delta} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} | f(b_n t) - f(x) | \, dt \right) p_{n,k}\left(\frac{x}{b_n}\right),$$

$$< \epsilon (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} dt \right) p_{n,k}\left(\frac{x}{b_n}\right) = \epsilon.$$

To calculate the value of I_2, we put $u = xb_n^{-1}$, and have

$$I_2 = (n+1) \sum_{|b_n t - x| \geqslant \delta} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} | f(b_n t) - f(x) | \, dt \right) p_{n,k}\left(\frac{x}{b_n}\right)$$

$$\leqslant 2M(n+1) \sum_{|t-u| \geqslant \delta/b_n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} dt \right) p_{n,k}(u)$$

$$\leqslant 2M(n+1)\left(\frac{\delta}{b_n}\right)^{-2} \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t-u)^2 \, dt \right) p_{n,k}(u)$$

$$\leqslant 2M\left(\frac{\delta}{b_n}\right)^{-2} \frac{u(1-u)}{n}$$

$$\leqslant 2M \frac{\frac{x}{b_n}}{n\left(\frac{\delta}{b_n}\right)^2},$$

for all large n, since $b_n = o(n)$.

Hence we have

$$| P_n(x) - f(x) | \leqslant \epsilon + \epsilon = 2\epsilon.$$

This completes the proof of theorem (1.3).

To prove Theorem 1.4, we need the following lemma :

Lemma — If $0 \leqslant x \leqslant 1$, the inequality

$$0 \leqslant z \leqslant \frac{3}{2}\left(\frac{x(1-x)}{n}\right)^{1/2} \qquad \ldots(2.2)$$

implies

$$\sum_{|t-x| \geqslant 2z(x(1-x)/n)^{1/2}} \int_{k/(n+1)}^{(k+1)/(n+1)} p_{n,k}(x) \, dt \leqslant 2e^{-z^2}.$$

Proof of Theorem 1.4: Proceeding in a similar manner as above, we obtain (2.1), as in Theorem 1.3,

$$| P_n(x) - f(x) |$$

$$\leqslant \epsilon + 2M(b_n)(n+1) \sum_{|t-u| \geqslant \delta/b_n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} dt \right) p_{n,k}(u).$$

The second term can be easily estimated by means of the above lemma, if

$$z = \delta n(2b_n)^{-1} \left[\frac{u(1-u)}{n}\right]^{-(1/2)},$$

the condition (2.2) is satisfied if we assume, for intance, that $\delta < 2x$ and that n is sufficiently large. Hence, by (1.1), we obtain

$$| P_n(x) - f(x) | \leqslant \epsilon + 2M(b_n) \exp(-z)^2$$

$$= \epsilon + 2M(b_n) \exp\{-\delta^2 . n [4b_n x(1 - xb_n^{-1})]^{-1}\},$$

$$\leqslant \epsilon + \epsilon = 2\epsilon,$$

for all large n.

This completes the proof of Theorem 1.4.

References

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