UNSTEADY FLOW OF A DUSTY VISCOUS FLUID THROUGH A UNIFORM PIPE WITH SECTOR OF A CIRCLE AS CROSS-SECTION

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The unsteady unidirectional flow of an incompressible viscous fluid with uniform distribution of dust particles through a uniform pipe whose cross-section is a sector of a circle subtending an angle 2α at the centre has been investigated. Analytical expressions for the velocity of the fluid and that of the dust particles under the influence of a constant pressure gradient have been derived. The influence of the dust particles on the fluid motion has been discussed and it is observed that the velocity of the dusty fluid is greater than that of the dust particles, but less than that of clean fluid.

INTRODUCTION

In recent years, workers in the field of fluid dynamics have been paying increasing attention to the study of the influence of dust particles on viscous fluid flows. Saffman (1962) has given the equations describing the motion of a gas containing small dust particles. Using these equations, Michael and Miller (1966) have discussed the motion of a dusty gas occupying the semi-infinite space above a rigid plane boundary. Sambasiva Rao (1969) has studied the unsteady flow of a dusty fluid through a uniform pipe under the influence of exponential pressure gradient with respect to time. With a similar pressure gradient, Reddy (1972) investigated the case of a rectangular channel. Girishwar Nath (1970) studied the unsteady flow of a dusty fluid between two rotating co-axial cylinders under the influence of an axial pressure gradient. In this paper, the authors have studied the unsteady motion of a dusty fluid. The dust is represented by a large number density N of small dust particles whose volume concentration is small, but has appreciable mass concentration. It is assumed that the individual particles of dust are so small that a Stokes flow approximation to their motion relative to the fluid is appropriate. The equations of motion give rise to two additional independent parameters due to the presence of the dust, which may be, as the mass concentration of the dust l and relaxation time τ. The latter parameter is representative of the time scale on which the velocity of the dust adjusts to changes in the neighbouring fluid velocity. When $\tau = 0$, this adjustment is instantaneous, and we have a limiting case in which the dust moves with fluid at each point. In our solutions, if the semi-angle α of the sector becomes π, the resulting solutions do not give exactly the flow as in the case of uniform pipe, because in this case there is a semi-diametral wall extending along the length of the pipe and joining a generator with the axis of the pipe.

EQUATIONS

We take for reference frame a cylindrical polar system of coordinates (r, θ, z), the z-axis being taken along the length of the pipe through which the flow is to be considered, the cross-section of the pipe being a sector bounded by two radii $\theta = \pm \alpha$ and the circle $r = a$. For the present problem, the velocity distributions of the fluid and dust particles are defined respectively as

$$u'_r = 0,\ u'_\theta = 0,\ u'_z = u'_z(r', \theta, t')$$

$$v'_r = 0,\ v'_\theta = 0,\ v'_z = v'_z(r', \theta, t')$$

where (u'_r, u'_θ, u'_z) and (v'_r, v'_θ, v'_z) are respectively the velocity components of the fluid and the dust particles.

The equations of motion given by Saffman (1962), for the present geometry, reduce to

$$\frac{\partial u'_z}{\partial t'} = -\frac{1}{\rho}\frac{\partial p'}{\partial z'} + \nu\left(\frac{\partial^2 u'_z}{\partial r'^2} + \frac{1}{r'}\frac{\partial u'_z}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2 u'_z}{\partial \theta^2}\right) + \frac{KN_0}{\rho}(v'_z - u'_z) \qquad \ldots(1)$$

and

$$\tau \frac{\partial v'_z}{\partial t'} = (u'_z - v'_z) \qquad \ldots(2)$$

where $\tau = m/K$.

Now, we introduce the following non-dimensional quantities :

$$w_1 = \frac{u'_z a}{\nu},\ w_2 = \frac{v'_z a}{\nu},\ p = \frac{p' a^2}{\rho \nu^2},\ z = \frac{z'}{a}$$

$$r = \frac{r'}{a},\ t = \frac{\nu t'}{a^2},\ l = \frac{mN_0}{\rho},\ \sigma = \frac{m\nu}{Ka^2}.$$

Equations (1) and (2) then become

$$\frac{\partial w_1}{\partial t} = -\frac{\partial p}{\partial z} + \left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right) + \frac{l}{\sigma}(w_2 - w_1), \qquad \ldots(3)$$

$$\frac{\partial w_2}{\partial t} = \frac{1}{\sigma}(w_1 - w_2). \qquad \ldots(4)$$

Differentiating (3) with respect to t once and using (4), we have

$$\frac{\partial w_1}{\partial t^2} = \frac{\partial}{\partial t}\left(-\frac{\partial p}{\partial z}\right) + \frac{\partial}{\partial t}\left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right) - \left(\frac{l+1}{\sigma}\right)\frac{\partial w_1}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\sigma}\left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right). \quad ...(5)$$

Solution

Let $-\dfrac{\partial p}{\partial z} = C$ (a constant for $t > 0$). Then, eqn. (5) becomes

$$\frac{\partial^2 w_1}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right) - \left(\frac{l+1}{\sigma}\right)\frac{\partial w_1}{\partial t} + \frac{C}{\sigma} + \frac{1}{\sigma}\left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right). \quad ...(6)$$

The no-slip boundary conditions are

$$w_1(r, \theta, t) = w_2(r, \theta, t) = 0, \text{ when } r = 0 \text{ and } r = 1,$$
$$t > 0, \ -\alpha \leqslant \theta \leqslant \alpha$$
$$w_1(r, \theta, t) = w_2(r, \theta, t) = 0, \text{ when } \theta = \pm\alpha, t > 0, 0 < r < 1. \quad ...(7a)$$

The initial conditions are

$$w_1(r, \theta, t) = w_2(r, \theta, t) = 0, \text{ when } t = 0, 0 \leqslant r \leqslant 1$$

and $-\alpha \leqslant \theta \leqslant \alpha.$ ...(7b)

Let us assume the solution as

$$w_1(r, \theta, t) = \sum_{n=1}^{\infty} \{A_{k_n}(r, t) \cos k_n\theta + B_{k_n}(r, t) \sin k_n\theta\} \quad ...(8)$$

where k_n is some function of the integer n.

With the help of (7), we can choose the transform as

$$\bar{w}_1 = \bar{w}_1(r, m, t) = \int_0^{\alpha} w_1(r, \theta, t) \cos \frac{2m+1}{2\alpha}\pi\theta \, d\theta. \quad ...(9)$$

Applying it to eqn. (6), we get

$$\frac{\partial^2 \bar{w}_1}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial^2 \bar{w}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}_1}{\partial r} - \frac{\mu^2}{r^2}\bar{w}_1\right) - \left(\frac{l+1}{\sigma}\right)\frac{\partial \bar{w}_1}{\partial t} + \frac{C}{\sigma}\frac{(-1)^m}{\mu} + \frac{1}{\sigma}\left(\frac{\partial^2 \bar{w}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}_1}{\partial r} - \frac{\mu^2}{r^2}\bar{w}_1\right), \quad \ldots(10)$$

where

$$\mu = (2m+1)\pi/2\alpha.$$

Using the finite Hankel transform technique (Sneddon 1951), eqn. (10) can be converted into

$$\frac{d^2 \bar{\bar{w}}_1}{dt^2} + \left(\xi_i^2 + \frac{l+1}{\sigma}\right)\frac{d\bar{\bar{w}}_1}{dt} + \frac{\xi_i^2 \bar{\bar{w}}_1}{\sigma} = \frac{C}{\sigma}\frac{(-1)^m}{\mu}F(\xi_i), \quad \ldots(11)$$

where

$$\bar{\bar{w}}_1 = \int_0^1 \bar{w}_1(r, m, t)\, r\, J_\mu(r\xi_i)\, dr,$$

and ξ_i, $i = 1, 2, 3, \ldots$, are the positive roots of the equation $J_\mu(\xi) = 0$.

The solution of (11) satisfying the corresponding boundary conditions is

$$\bar{\bar{w}}_1 = \left[\frac{m_2 e^{m_1 t} - m_1 e^{m_2 t}}{m_1 - m_2} + 1\right]\frac{C}{\xi_i^2}\frac{(-1)^m}{\mu}F(\xi_i), \quad \ldots(12)$$

where

$$m_1, m_2 = \tfrac{1}{2}\left[-\left(\xi_i^2 + \frac{l+1}{\sigma}\right) \pm \sqrt{\left(\xi_i^2 + \frac{l+1}{\sigma}\right)^2 - 4\frac{\xi_i^2}{\sigma}}\right].$$

Applying inversion formula for the Hankel transform to eqn. (12), we get

$$\bar{w}_1(r, m, t) = 2\sum_{i=1}^{\infty}\left[\frac{m_2 e^{m_1 t} - m_1 e^{m_2 t}}{m_1 - m_2} + 1\right] \times \frac{C}{\xi_i^2}\frac{(-1)^m}{\mu}F(\xi_i)\frac{J_\mu(r\xi_i)}{[J'_\mu(\xi_i)]^2}. \quad \ldots(13)$$

Using the inversion formula for the cosine transform to eqn. (13), we have

$$w_1(r, \theta, t) = \frac{4C}{\alpha}\sum_{m=0}^{\infty}\frac{(-1)^m}{\mu}\sum_{i=1}^{\infty}\left[\frac{m_2 e^{m_1 t} - m_1 e^{m_2 t}}{m_1 - m_2} + 1\right] \times$$

(*equation continued on p.* 700)

$$\frac{F(\xi_i)}{\xi_i^2} \frac{J_\mu(r\xi_i)}{[J'_\mu(\xi_i)]^2} \cos(2m+1)\pi\theta/2\alpha. \qquad \ldots(14)$$

From (14) and (4), we have

$$w_2 = A(r,\theta)\, e^{-(1/\sigma)t} + \frac{4C}{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{\mu}$$

$$\times \sum_{i=1}^{\infty} \left[\frac{m_2 e^{m_1 t}}{\sigma m_1 + 1} - \frac{m_1 e^{m_2 t}}{\sigma m_2 + 1} + (m_1 - m_2) \right]$$

$$\times \frac{F(\xi_i)}{\xi_i^2} \frac{J_\mu(r\xi_i)}{[J'_\mu(\xi_i)]^2} \cos(2m+1)\pi\theta/2\alpha \qquad \ldots(15)$$

where $A(r, \theta)$ is a constant to be determined with the help of the initial conditions.

Thus, we get

$$w_2(r,\theta,t) = \frac{4C}{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{\mu}$$

$$\times \sum_{i=1}^{\infty} \left[\frac{m_2(\sigma m_2+1)e^{m_1 t} - m_1(\sigma m_1+1)e^{m_2 t}}{(\sigma m_1+1)(\sigma m_2+1)(m_1-m_2)} - \frac{\sigma^2 m_1 m_2 e^{-(1/\sigma)t}}{(\sigma m_1+1)(\sigma m_2+1)} + 1 \right]$$

$$\times \frac{F(\xi_i)}{\xi_i^2} \frac{J_\mu(r\xi_i)}{[J'_\mu(\xi_i)]^2} \cos(2m+1)\pi\theta/2\alpha. \qquad \ldots(16)$$

The expression for w, the velocity of the clean fluid, is

$$w(r,\theta,t) = \frac{4C}{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{\mu} \sum_{i=1}^{\infty} \left[1 - e^{-\xi_i^2 t} \right]$$

$$\times \frac{F(\xi_i)}{\xi_i^2} \frac{J_\mu(r\xi_i)}{[J'_\mu(\xi_i)]^2} \cos(2m+1)\pi\theta/2\alpha. \qquad \ldots(17)$$

From (14) and (17), one can easily observe that the velocity of the clean fluid is greater than that of the dusty fluid. It means that dust particles reduce the velocity of the clean fluid. The dependence of axial velocities on θ is obvious. The axial velocity components will be zero on the two walls ($\theta = \pm \alpha$) and the semi-vertical angle of the pipe will have to be adequately large ($r\alpha >$ the viscous layer) to allow the axial velocity to fully develop to the maximum value at $\theta = 0$.

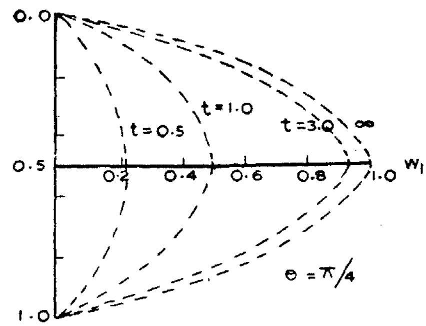

FIG. 1. Velocity profiles for fluid particles.

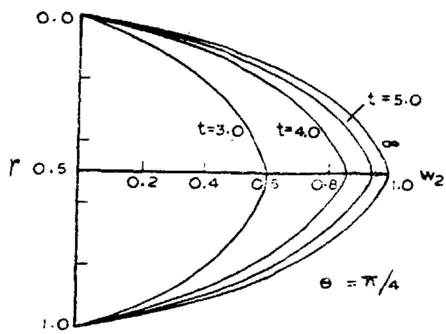

FIG. 2. Velocity profiles for dust particles.

The change in the velocity profiles with time is shown in Figs. 1 and 2. Initially, both the fluid and the dust particles are at rest. It is clear that w_1 and w_2 increase with t for fixed r and θ. The velocity of the fluid at any point in the flow field is greater than that of the dust particles. The velocity of the fluid and that of the dust particles become the same, when $\sigma \to 0$, i.e. when the dust particles become very fine. It means that if the masses of dust particles are small, their influence on the flow is reduced and in the limit $m \to 0$, the fluid becomes ordinary viscous.

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