DIFFRACTION BY A UNI-DIRECTIONALLY CONDUCTING STRIP

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The problem of diffraction of a plane electromagnetic wave by a uni-directionally conducting infinite strip has been solved by a method due to Jones (1964) involving solution of the corresponding problem in the case of a perfectly conducting strip, based on the Wiener-Hopf technique. The problem has been reduced to a pair of coupled integral equations and approximate solutions of these integral equations have been obtained up to $O(l^{-3/2})$, for large l, where l stands for the width of the strip. The far-field has been evaluated by the method of steepest descent.

1. INTRODUCTION

A uni-directionally conducting surface is the one on which the current is constrained to flow in one direction, but is otherwise perfectly conducting (cf. Jones 1964; Hurd 1960). The problem of diffraction of a plane electromagnetic wave by a uni-directionally conducting semi-infinite screen has been solved by Hurd (1960) and Karp (1957) by the application of transform techniques. The problem of diffraction of a dipole field by a uni-directionally conducting semi-infinite screen has been solved by Radlow (1957) by the application of double Laplace transforms and modified Wiener-Hopf technique.

The Wiener-Hopf technique serves as a powerful weapon for attacking two-dimensional diffraction problems. A good account of the technique has been given by Noble (1958) and Jones (1964). As application of the Wiener-Hopf technique, various authors have considered the problems of diffraction by perfectly conducting screens (Noble 1958; Jones 1964). Senior (1952) solved the problem of diffraction of plane electromagnetic waves by an imperfectly conducting semi-infinite sheet, where impedance-type boundary conditions have been employed on the faces of the sheet. Faulkner (1965) solved the problem of diffraction by a metallic strip, using the technique of Jones, employing impedance boundary conditions similar to those employed by Senior (1952). There has been considerable progress in the application of the Wiener-Hopf technique in recent years and problems concerning diffraction in anisotropic media have also been solved by Seshadri and Rajagopal (1963), Williams (1965) and others, using this technique.

In the present paper, we have investigated the problem of diffraction of a plane electromagnetic wave by a uni-directionally conducting strip of width l. We have reduced the problem to that of solving a pair of coupled integral equations by the application of the Wiener-Hopf technique and Jones' method. These integral equations involve the parameter l and they have been solved approximately by a technique due to Jones (1964, pp. 602-607), up to the terms of order $l^{-3/2}$, where $l \gg 1$. The far field has been evaluated applying the method of steepest descent.

2. Formulation of the Problem

A uni-directionally conducting strip occupies the region $z = 0$, $-l \leqslant x < 0$ of the (xyz)-space. We employ a second rectangular coordinate system (ξ, η, z) by means of the relations :

$$x = \xi \cos \alpha - \eta \sin \alpha, \quad y = \xi \sin \alpha + \eta \cos \alpha, \quad z = z, \qquad \ldots(2.1)$$

where $\alpha(-\pi/2 \leqslant \alpha \leqslant \pi/2)$ is the angle between the positive x and the positive ξ directions.

We assume that the strip is infinitely conducting in the ξ-direction and the conductivity is taken to be zero in the η direction.

Upon the strip there is an incident plane electromagnetic wave whose electric vector is given by

$$\vec{E}^{(i)} = \vec{A} \exp \{i(-\vec{K}\,\vec{r} + \omega t) \qquad \ldots(2.2)$$

where $\vec{K} = (K_1, K_2, K_3)$ and ω is the frequency

It is required to find the total electromagnetic fields $\vec{E}$ and $\vec{H}$ which satisfy the following boundary conditions (cf. Faulkner 1965, p. 655) :

$$E_\xi = 0 \text{ on the strip} \qquad \ldots(2.3)$$

$$E_\eta \text{ is continuous across the strip} \qquad \ldots(2.4)$$

$$H_\xi \text{ is continuous across the strip.} \qquad \ldots(2.5)$$

In addition to the boundary conditions (2.3) – (2.5), the edge conditions of Jones (1964, pp. 566-569) and Radlow (1957) demand, for the uniqueness of the solution, that the various field components have singularities of order less than or equal to $\rho^{-1/2}$ at the edges of the strip, where ρ is the distance measured from the edges.

Denoting the scattered electric field by $\vec{E}^{(s)}$, the total field $\vec{E}$ is given by

$$\vec{E} = \vec{E}^{(i)} + \vec{E}^{(s)}. \qquad \ldots(2.6)$$

We assume that the scattered fields can be derived from the one-component electric Hertz vector (cf. Jones 1964) :

$$\vec{\pi}^{(s)} = (\pi_\xi^{(s)}, 0, 0), \text{ in } (\xi, \eta, z) \text{ co-ordinate system.} \quad ...(2.7)$$

Then, from Jones (1964), we observe that the scattered fields are given by (the free space is assumed to be free from charges and currents) :

$$\left.\begin{aligned} \vec{E}^{(s)} &= \vec{\nabla} \times \vec{\nabla} \times \vec{\pi}^{(s)} \\ \vec{H}^{(s)} &= iKC\epsilon_0(\vec{\nabla} \times \vec{\pi}^{(s)}) \end{aligned}\right\} \quad ...(2.8)$$

where $K = \omega/C$; $C = (\mu_0\epsilon_0)^{-1/2}$; ϵ_0, μ_0 being the free space parameters.

We have dropped the time factor $e^{i\omega t}$ in (2.8) and we shall drop this factor throughout the rest of our analysis.

The scalar $\pi_\xi^{(s)}$ satisfies the reduced wave equation :

$$\frac{\partial^2\pi_\xi^{(s)}}{\partial x^2} + \frac{\partial^2\pi_\xi^{(s)}}{\partial y^2} + \frac{\partial^2\pi_\xi^{(s)}}{\partial z^2} + K^2\pi_\xi^{(s)} = 0. \quad ...(2.9)$$

Assuming $\pi_\xi^{(s)} = P_\xi^{(s)}(x, z)\, e^{-iK_2 y}$ ...(2.10)

eqn. (2.9) becomes

$$\frac{\partial^2 P_\xi^{(s)}}{\partial x^2} + \frac{\partial^2 P_\xi^{(s)}}{\partial z^2} + \beta^2 P_\xi^{(s)} = 0 \quad ...(2.11)$$

where $\beta^2 = K^2 - K_2^2$.

By means of eqns. (2.8), (2.10) and (2.11), we obtain the following field components of our interest :

$$e^{iK_2 y} E_\xi^{(s)} = -\left[\frac{\partial^2 P_\xi^{(s)}}{\partial x^2}\sin^2\alpha - K_2^2\cos^2\alpha\, P_\xi^{(s)} + iK_2\sin 2\alpha\frac{\partial P_\xi^{(s)}}{\partial x} + \frac{\partial^2 P_\xi^{(s)}}{\partial z^2}\right] \quad ...(2.12)$$

$$e^{iK_2 y} H_\eta^{(s)} = iKC\epsilon_0\frac{\partial P_\xi^{(s)}}{\partial z} \quad ...(2.13)$$

$$e^{iK_2 y} E_\eta^{(s)} = -\left[\left\{\frac{\partial^2 P_\xi^{(s)}}{\partial x^2} + K_2^2 P_\xi^{(s)}\right\} \sin\alpha\cos\alpha \right.$$
$$\left. + iK_2 \cos 2\alpha \,.\, \frac{\partial P_\xi^{(s)}}{\partial x}\right] \qquad ...(2.14)$$

and $$H_\xi^{(s)} = 0. \qquad ...(2.15)$$

We observe that the boundary condition (2.5) is automatically satisfied by our choice of the Hertz vector.

The problem is thus reduced to finding the solution of the partial differential equation (2.11) under the boundary conditions :

(i) $$E_\xi^{(s)} = -A_\xi e^{-iK_1 x - iK_2 y}, \text{ on } z = 0,\ -l < x < 0, \qquad ...(2.16)$$

(ii) $$E_\eta^{(s)}(0^+) = E_\eta^{(s)}(0^-), \text{ on } z = 0, \text{ for all } x, \qquad ...(2.17)$$

and appropriate edge-conditions at the edges $x = -l$ and $x = 0$ (cf. Karp 1957), where $E_\xi^{(s)}$ and $E_\eta^{(s)}$ are given by (2.12) and (2.14) in terms of $P_\xi^{(s)}$.

3. Reduction to Integral Equations

We employ the bilateral Laplace transform of the function $P_\xi^{(s)}$ denoted by

$$\Phi(s, z) = \int_{-\infty}^{\infty} P_\xi^{(s)}(x, z)\, e^{-sx}\, dx. \qquad ...(3.1)$$

We write

$$\Phi(s, z) = \Phi_1(s, z) + \Phi^+(s, z) + \Phi^-(s, z) \qquad ...(3.2)$$

where

$$\left.\begin{aligned} \Phi^+(s, z) &= \int_0^\infty P_\xi^{(s)}\, e^{-sx}\, dx \\ \Phi^-(s, z) &= \int_{-\infty}^{-l} P_\xi^{(s)}\, e^{-sx}\, dx \\ \text{and} \quad \Phi_1(s, z) &= \int_{-l}^{0} P_\xi^{(s)}\, e^{-sx}\, dx. \end{aligned}\right\} \qquad ...(3.3)$$

The functions Φ^+ and Φ^- are analytic in some right and left half-planes, denoted by Re $(s) > -\delta$ and Re $(s) < \delta$ respectively, whereas the function Φ_1 is an integral function of s. We have assumed that β has a small negative imaginary part, which will be put zero at the end.

We note that (see Jones 1964, p. 603) as $|s| \to \infty$ in the negative half-plane, $\Phi_1(s, z)$ will be $O(1)$, whereas $\Phi^-(s, z)$ will be $O(e^{sl})$.

When the appropriate edge conditions are applied [see Radlow (1957), and Jones (1964), p. 569 and 576], we find that

(e_1) $\Phi^+(s, 0) \sim O\left(\frac{1}{s^{5/2}}\right)$, as $|s| \to \infty$ in the positive half-plane,

(e_2) $\Phi^-(s, 0) \sim O\left(\frac{e^{sl}}{s^{5/2}}\right)$, as $|s| \to \infty$ in the negative half-plane,

(e_3) $\Phi^{+''}(s, 0) \sim O(s^{-1/2})$, as $|s| \to \infty$ in the positive half-plane,

(e_4) $\Phi^{-''}(s, 0) \sim O\left(\frac{e^{sl}}{s^{1/2}}\right)$, as $|s| \to \infty$ in the negative half-plane,

(e_5) $\Phi^{+'}(s, 0) \sim O\left(\frac{1}{s^{3/2}}\right)$, as $|s| \to \infty$ in the positive half-plane,

and (e_6) $\Phi^{-'}(s, 0) \sim O\left(\frac{e^{sl}}{s^{3/2}}\right)$, as $|s| \to \infty$ in the negative half-plane,

where dashes denote derivatives with respect to z.

Employing the transform (3.1) to the partial differential equation (2.11) and using the condition (2.17), we find that the appropriate solution of the transformed equation, satisfying the infinity condition, is given by

$$\Phi(s, z) = A(s)\, e^{-i\kappa |z|} \qquad \text{...(3.5)}$$

where $\kappa = (s^2 + \beta^2)^{1/2}$, assuming that β and K have small negative imaginary parts which will be put equal to zero at the end. (Note that $\beta^2 = K_1^2 + K_3^2$).

Considering the limiting values of the function $\Phi(s, z)$ and its derivatives, on the boundary $z = 0$, and using the conditions of continuity of the function and its second order derivative on $z = 0$ [as suggested by (2.16) and (2.17)] we find that

$$A(s) = \Phi^-(s, 0) + \Phi_1(s, 0) + \Phi^+(s, 0) \qquad \text{...(3.6)}$$

$$-\kappa^2 A(s) = \Phi^{-''}(s, 0) + \Phi_1''(s, 0) + \Phi^{+''}(s, 0) \qquad \text{...(3.7)}$$

and

$$-2i\kappa A(s) = \Phi_1'(s, 0^+) - \Phi_1'(s, 0^-). \qquad \text{...(3.8)}$$

The boundary condition (2.16) with (2.12) gives

$$\Phi_1''(s, 0) + (s \sin \alpha - iK_2 \cos \alpha)^2 \Phi_1(s, 0)$$
$$= \frac{B_\xi}{s + iK_1} [1 - \exp \{(s + iK_1)l\}] \qquad ...(3.9)$$

where $B_\xi = - A_\xi$.

Multiplying (3.6) by $(s \sin \alpha - iK_2 \cos \alpha)^2$, adding the result to (3.7) and using (3.9), we obtain with the help of (3.8) [thus eliminating $A(s)$ from the above equations using (3.9)]:

$$\Psi^+(s, 0) + \Psi^-(s, 0) + \frac{B_\xi}{(s + iK_1)} [1 - \exp \{(s + iK_1) l\}]$$
$$= - \frac{i}{2\kappa} P(s) I_2(s), \qquad ...(3.10)$$

where

$$\left.\begin{aligned} \Psi^+(s, 0) &= \Phi^{+''}(s, 0) + (s \sin \alpha - iK_2 \cos \alpha)^2 \Phi^+(s, 0) \\ \Psi^-(s, 0) &= \Phi^{-''}(s, 0) + (s \sin \alpha - iK_2 \cos \alpha)^2 \Phi^-(s, 0) \\ I_2(s) &= \Phi_1'(s, 0^+) - \Phi_1'(s, 0^-), \\ \text{and} \quad P(s) &= (s \sin \alpha - iK_2 \cos \alpha)^2 - \kappa^2. \end{aligned}\right\} \qquad ...(3.11)$$

We observe that

$$P(s) = - \cos^2 \alpha \, (s - ic^+)(s - ic^-) \qquad ...(3.12)$$

where

$$\left.\begin{aligned} c^+ &= - K \sec \alpha - K_2 \tan \alpha \\ \text{and} \quad c^- &= K \sec \alpha - K_2 \tan \alpha. \end{aligned}\right\} \qquad ...(3.13)$$

It can be shown that (under the assumption on the imaginary parts of K and β)

$$\text{Re}\,(ic^+) \leqslant \text{Re}\,(- i\beta), \;\; \text{Re}\,(ic^-) \geqslant \text{Re}\,(i\beta), \;\; \text{for} \; - \frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}. \qquad ...(3.14)$$

Thus, we note that $(s - ic^+)^{-1}$ is analytic in the positive half-plane $\text{Re}\,(s) > \text{Re}\,(- i\beta)$ and $(s - ic^-)^{-1}$ is analytic in the negative half-plane $\text{Re}\,(s) < \text{Re}\,(i\beta)$.

Multiplying both sides of eqn. (3.10), once by $\dfrac{(s + i\beta)^{1/2}}{(s - ic^+)}$ and another time by $\dfrac{(s - i\beta)^{1/2}}{(s - ic^-)} . e^{-sl}$, we obtain the following two Wiener-Hopf equations for the determination of the functions $\Psi^+(s, 0)$, $\Psi^-(s, 0)$ and $I_2(s)$:

$$\frac{(s+i\beta)^{1/2}}{(s-ic^+)}\Psi^+(s,0)+\frac{(s+i\beta)^{1/2}}{(s-ic^+)}\left[\Psi^-(s,0)+\frac{B_\xi}{s+iK_1}\{1-\exp(s+iK_1)\,l\}\right]$$
$$=-\frac{i\cos^2\alpha}{2(s-i\beta)^{1/2}}\cdot(s-ic^-)\,I_2(s) \quad \ldots(A)$$

and

$$\Bigg(\frac{(s-i\beta)^{1/2}}{(s-ic^-)}\Psi^-(s,0)+\frac{(s-i\beta)^{1/2}}{(s-ic^-)}\Bigg[\Psi^+(s,0)+\frac{B_\xi}{s+iK_1}$$
$$\times\Big\{1-\exp(s+iK_1)\,l\Big\}\Bigg]\Bigg)e^{-sl}$$
$$=-\frac{i\cos^2\alpha}{2(s+i\beta)^{1/2}}(s-ic^+)\,I_2(s)\,e^{-sl}. \quad \ldots(B)$$

Equations (A) and (B) are satisfied for $\mathrm{Re}\,(-i\beta)<\mathrm{Re}\,(s)<\mathrm{Re}\,(i\beta)$. Employing the splitting technique (see Jones 1964, p. 603)

$$X(s)=X_+(s)+X_-(s),\ [X(s)\text{ analytic in }-\delta<\mathrm{Re}\,(s)<\delta]$$

where

$$X_+(s)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{X(\omega)}{\omega-s}\,d\omega,\ (\mathrm{Re}\,(s)>a)$$

and

$$X_-(s)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{X(\omega)}{\omega-s}\,d\omega,\ \ (\mathrm{Re}\,(s)<b),\qquad(-\delta<a<b<\delta)$$

and using the conditions (3.4) and the extended Liouville's theorem, we obtain (as in Jones 1964, p. 603)

$$\left.\begin{aligned}
&\frac{(s+i\beta)^{1/2}}{(s-ic^+)}\Psi^+(s,0)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(\omega+i\beta)^{1/2}\,d\omega}{(\omega-ic^+)(\omega-s)}\\
&\qquad\times\left[\Psi^-(\omega,0)+\frac{B_\xi}{\omega+iK_1}\{1-\exp(\omega+iK_1)\,l\}\right]\\
&\text{and}\\
&\frac{(s-i\beta)^{1/2}}{(s-ic^-)}\Psi^-(s,0)\,e^{-sl}=-\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{(\omega-i\beta)^{1/2}\,e^{-\omega l}\,d\omega}{(\omega-ic^-)(\omega-s)}\\
&\qquad\times\left[\Psi^+(\omega,0)+\frac{B_\xi}{\omega+iK_1}\{1-\exp(\omega+iK_1)l\}\right]
\end{aligned}\right\}\ldots(3.15)$$

where $\quad -\delta<a<b<\delta,\ \ \delta=\mathrm{Re}\,(i\beta).$

We now change the variable of integration in the first of eqns. (3.15) from ω to $-\omega$ putting $b = -a$, and change s in the second of eqns. (3.15) to $-s$, and obtain : [using that branch of the square-root for which $(-s + i\beta)^{1/2} = i(s - i\beta)^{1/2}$]:

$$(s + i\beta)^{1/2} F^+(s) = \frac{1}{2\pi} \int_{b-i\infty}^{b+i\infty} d\omega \frac{(\omega - i\beta)^{1/2}}{\omega + s} \left[\frac{\omega + ic^-}{\omega + ic^+} \omega^+(\omega) - \frac{B_\xi\{1 - \exp(-\omega + iK_1) l\}}{(\omega + ic^+)(\omega - iK_1)} \right] \quad ...(3.16)$$

and

$$e^{sl}(s + i\beta)^{1/2} G^+(s) = \frac{1}{2\pi} \int_{b-i\infty}^{b+i\infty} d\omega \, . \frac{(\omega - i\beta)^{1/2}}{\omega + s} \left[\frac{\omega - ic^+}{\omega - ic^-} F^+(\omega) + \frac{B_\xi\{1 - \exp(\omega + iK_1)l\}}{(\omega - ic^-)(\omega + iK_1)} \right] e^{-\omega l}, \quad ...(3.17)$$

where

$$F^+(s) = \Psi^+ \frac{(s, 0)}{(s - ic^+)}, \quad G^+(s) = \Psi^- \frac{(-s, 0)}{(s + ic^-)}, \quad ...(3.18)$$

and

$$\text{Re}\,(ic^+) < a < b < \text{Re}\,(ic^-).$$

Equations (3.16) and (3.17) are the desired integral equations for the unknown functions F^+ and G^+, or Ψ^+ and Ψ^-.

Knowledge of these functions helps the determination of the single unknown $A(s)$ by (3.8) and (3.10).

Eqns. (3.16) and (3.17) can be cast into the most convenient forms, for their solution, in terms of the new unknowns :

$$\left.\begin{aligned} F^*(s) &= F^+(s) + B_\xi/\{(s + iK_1)(s - ic^+)\} \\ &\text{and} \\ G^*(s) &= G^+(s)\, e^{sl} + B_\xi e^{iK_1 l}/\{(s - iK_1)(s + ic^-)\}. \end{aligned}\right\} \quad ...(3.19)$$

Then, choosing $b = c$, where $\text{Re}\,(ic^+) < c < \text{Re}\,(ic^-)$, with $c > 0$, the integrals in (3.16) and (3.17), not involving the unknowns, can be evaluated by closing the contours to the left and we finally obtain the following integral equations, for the new unknowns $F^*(s)$ and $G^*(s)$:

$$(s + i\beta)^{1/2} F^*(s) = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{(\omega - i\beta)^{1/2}}{(\omega + s)} \cdot \left(\frac{\omega + ic^-}{\omega + ic^+} \right) G^*(\omega)\, e^{-\omega l}\, d\omega - \frac{B_\xi e^{i\pi/4} . (K_1 - \beta)^{1/2}}{(K_1 + c^+)(s + iK_1)} \quad ...(3.20)$$

and

$$(s+i\beta)^{1/2}G^*(s)=\frac{1}{2\pi}\int\limits_{c-i\infty}^{c+i\infty}\frac{(\omega-i\beta)^{1/2}}{(\omega+s)}\cdot\left(\frac{\omega-ic^+}{\omega-ic^-}\right)F^*(\omega)\,e^{-\omega l}\,d\omega$$
$$+\frac{B_\xi e^{iK_1 l-i\pi/4}.(K_1+\beta)^{1/2}}{(K_1+c^-)(s-iK_1)}. \qquad \ldots(3.21)$$

We note, for future necessity, from (3.19), (3.18), (3.10) and (3.8) that

$$P(s)\,A(s)=(s-ic^+)\,F^*(s)-(s-ic^-)\,e^{sl}G^*(-s). \qquad \ldots(3.22)$$

4. Approximate Solution of the Integral Equations (3.20) and (3.21) for $l \gg 1$

In this section, we shall obtain approximate solutions of the integral equations (3.20) and (3.21) by a technique similar to that of Jones (1964, pp. 607-609). We write :

$$\left.\begin{aligned}\mathcal{J}(s)&=\frac{1}{2\pi}\int\limits_{c-i\infty}^{c+i\infty}\frac{(\omega-i\beta)^{1/2}}{(\omega+s)}\cdot\left(\frac{\omega-ic^+}{\omega-ic^-}\right)F^*(\omega)\,e^{-\omega l}\,d\omega\\ &\text{and}\\ \mathcal{G}(s)&=\frac{1}{2\pi}\int\limits_{c-i\infty}^{c+i\infty}\frac{(\omega-i\beta)^{1/2}}{(\omega+s)}\cdot\left(\frac{\omega+ic^-}{\omega+ic^+}\right)G^*(\omega)\,e^{-\omega l}\,d\omega.\end{aligned}\right\} \qquad \ldots(4.1)$$

Then (3.20) and (3.21) give

$$\left.\begin{aligned}F^*(s)&=\left[\mathcal{G}(s)-\frac{B_\xi e^{i\pi/4}\,(K_1-\beta)^{1/2}}{(K_1+c^+)(s+iK_1)}\right](s+i\beta)^{-1/2}\\ &\text{and}\\ G^*(s)&=\left[\mathcal{J}(s)+\frac{B_\xi e^{iK_1 l-i\pi/4}.(K_1+\beta)^{1/2}}{(K_1+c^-)(s-iK_1)}\right](s+i\beta)^{-1/2}.\end{aligned}\right\} \qquad \ldots(4.2)$$

Substituting (4.2) in (4.1), we obtain

$$\mathcal{J}(s)=-\frac{B_\xi e^{i\pi/4}\,(K_1-\beta)^{1/2}}{(K_1+c^+)}\cdot\frac{1}{2\pi}\int\limits_{c-i\infty}^{c+i\infty}\left(\frac{\omega-i\beta}{\omega+i\beta}\right)^{1/2}$$
$$\times\frac{(\omega-ic^+)\,e^{-\omega l}\,d\omega}{(\omega-ic^-)(\omega+iK_1)(\omega+s)}$$
$$+\frac{1}{2\pi}\int\limits_{c-i\infty}^{c+i\infty}\left(\frac{\omega-i\beta}{\omega+i\beta}\right)^{1/2}\left(\frac{\omega-ic^+}{\omega-ic^-}\right)\cdot\frac{\mathcal{G}(\omega)\,e^{-\omega l}}{(\omega+s)}\,d\omega, \qquad \ldots(4.3)$$

and

$$\mathcal{G}(s) = \frac{B_g e^{iK_1 l - i\pi/4}}{(K_1 + c^-)} (K_1 + \beta)^{1/2} \cdot \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \left(\frac{\omega - i\beta}{\omega + i\beta}\right)^{1/2} \times \frac{(\omega + ic^-)\, e^{-\omega l}\, d\omega}{(\omega + ic^+)(\omega - iK_1)(\omega + s)}$$

$$+ \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \left(\frac{\omega - i\beta}{\omega + i\beta}\right)^{1/2} \cdot \left(\frac{\omega + ic^-}{\omega + ic^+}\right) \cdot \frac{\mathcal{J}(\omega)\, e^{-\omega l}}{(\omega + s)}\, d\omega. \qquad \text{...(4.4)}$$

Now noting that $\mathcal{J}(s)$ and $\mathcal{G}(s)$ are analytic in the positive half-plane $\mathrm{Re}\,(s) > c$, the integrals involving $\mathcal{J}(\omega)$ and $\mathcal{G}(\omega)$ in (4.3) and (4.4) can be evaluated by closing the contours to the right, with a loop around the branch-cut from $i\beta$ to ∞, and by taking care of the poles at $\omega = ic^-$ and $\omega = ic^+$ for the integrals containing $\mathcal{G}(\omega)$ and $\mathcal{J}(\omega)$ respectively. The integrals around the branch-cut can be evaluated for large l by using Watson's lemma and by means of the result [see Jones 1964, p. 606]:

$$\frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{(\omega - i\beta)^{1/2}}{\omega + s} e^{-\omega l}\, d\omega = -(\pi l)^{-1/2} e^{-i\beta l} + e^{sl}(s + i\beta)^{1/2}\, \mathrm{erfc}\{(s + i\beta)\, l\}^{1/2}$$

$$\approx - \frac{e^{-i\beta l}}{2\pi^{1/2}(s + i\beta)} \cdot \frac{1}{l^{3/2}}, \text{ for } l \gg 1, \qquad \text{...(4.5)}$$

where the erfc has been approximated by means of the first two terms of the asymptotic expansion:

$$\mathrm{erfc}\,(z) \approx \left(\frac{e^{-z^2}}{\pi^{1/2} z}\right) [1 - \tfrac{1}{2} z^{-2}]. \qquad \text{...(4.6)}$$

This technique of approximate evaluation of the various integrals occurring in the present problem will be valid if $\alpha \neq 0$. In case, when $\alpha = 0$, the problem has to be reformulated and its solution is given in Jones (1964, p. 603-7).

Thus, by means of the above arguments, we find that for $l \gg 1$, and $\alpha \neq 0$,

$$\frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \left(\frac{\omega - i\beta}{\omega + i\beta}\right)^{1/2} \left(\frac{\omega + ic^-}{\omega + ic^+}\right) \cdot \frac{\mathcal{J}(\omega)\, e^{-\omega l}}{(\omega + s)}\, d\omega$$

$$\approx \left(\frac{c^- - c^+}{s - ic^+}\right) \left(\frac{c^+ + \beta}{c^+ - \beta}\right)^{1/2} \mathcal{J}(-ic^+) \exp\,(ic^+ l)$$

$$- (2\beta)^{-1/2} \left(\frac{c^- + \beta}{c^+ + \beta}\right) \frac{\mathcal{J}(i\beta)}{2\pi^{1/2}} \cdot \frac{e^{-i\beta l - i\pi/4}}{(s + i\beta)\, l^{3/2}}, \qquad \text{...(4.7)}$$

and

$$\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{\omega-i\beta}{\omega+i\beta}\right)^{1/2}\left(\frac{\omega-ic^+}{\omega-ic^-}\right)\frac{\mathcal{J}(\omega)\,e^{-\omega l}}{(\omega+s)}\,d\omega$$

$$\approx\left(\frac{c^--c^+}{s+ic^-}\right)\left(\frac{c^--\beta}{c^-+\beta}\right)^{1/2}\mathcal{J}(ic^-)\exp(ic^-l)$$

$$-(2\beta)^{-1/2}\left(\frac{\beta-c^+}{\beta-c^-}\right)\mathcal{J}(i\beta)\cdot\frac{\exp\{-i\beta l-i\pi/4\}}{2(s+i\beta)\,\pi^{1/2}\,l^{3/2}}. \quad \ldots(4.8)$$

The integrals in (4.3) and (4.4), not involving the unknowns, can also be evaluated by means of Watson's lemma and the result (4.5) after breaking the integrands into partial fractions.

We finally obtain, using (4.7) and (4.8) (to the order of $l^{-3/2}$):

$$\mathcal{I}(s)=-B_\xi\cdot\left\{\frac{(K_1-\beta)(c^--\beta)}{(c^-+\beta)}\right\}^{1/2}$$

$$\times\left\{\frac{(c^--c^-)}{(K_1+c^-)(K_1+c^+)(s+ic^-)}\right\}\exp\left(-\frac{i\pi}{4}-ic^-l\right)$$

$$+\left(\frac{c^--c^+}{s+ic^-}\right)\left(\frac{c^--\beta}{c^-+\beta}\right)^{1/2}\mathcal{J}(ic^-)\,e^{-ic^-l}$$

$$+\left\{\frac{-iB_\xi(K_1-\beta)^{1/2}}{(K_1+c^+)}\cdot\frac{(\beta-c^+)}{(\beta-c^-)(K_1+\beta)}\right.$$

$$\left.-e^{-i\pi/4}\cdot\left(\frac{\beta-c^+}{\beta-c^-}\right)\mathcal{J}(i\beta)\right\}\cdot\frac{(2\beta)^{-1/2}\,e^{-i\beta l}}{2\pi^{1/2}\,l^{3/2}\,(s+i\beta)}, \quad \ldots(4.9)$$

and

$$\mathcal{J}(s)=-B_\xi\left\{\frac{(K_1+\beta)(c^++\beta)}{(\beta-c^+)}\right\}^{1/2}$$

$$\times\left\{\frac{(c^--c^+)}{(K_1+c^-)(K_1+c^+)(s-ic^+)}\right\}\exp\{i(c^+l+K_1l+\pi/4)\}$$

$$+\left(\frac{c^--c^+}{s-ic^+}\right)\left(\frac{c^++\beta}{c^+-\beta}\right)^{1/2}\mathcal{I}(-ic^+)\,e^{ic^+l}$$

$$+\left\{\frac{B_\xi(K_1+\beta)^{1/2}}{(K_1+c^-)}\cdot\frac{(\beta+c^-)\,e^{iK_1l}}{(\beta+c^+)(\beta-K_1)}\right.$$

$$\left.-e^{-i\pi/4}\,\mathcal{I}(i\beta)\left(\frac{\beta+c^-}{\beta+c^+}\right)\right\}\cdot\frac{(2\beta)^{-1/2}\,e^{-i\beta l}}{2\pi^{1/2}\,l^{3/2}\,(s+i\beta)}. \quad \ldots(4.10)$$

It is thus observed that the functions $\mathcal{I}(s)$ and $\mathcal{J}(s)$ have been determined up to the terms of order $l^{-3/2}$ by means of eqns. (4.9) and (4.10) and they involve, once again, the unknown constants $\mathcal{I}(i\beta)$, $\mathcal{G}(i\beta)$, $\mathcal{I}(-ic^+)$ and $\mathcal{G}(ic^-)$. However, these constants can be determined up to this order $[O(l^{-3/2})]$ of accuracy by direct substitutions, in succession in (4.9) and (4.10).

We obtain

$$\mathcal{I}(i\beta) = B_\xi \left\{\frac{(K_1-\beta)(c^- - \beta)}{(\beta + c^-)}\right\}^{1/2} \times \left\{\frac{(c^- - c^+)}{(K_1 + c^-)(K_1 + c^+)(\beta + c^-)}\right\} \exp\left(\frac{i\pi}{4} - ic^- l\right) - i\left(\frac{c^- - c^+}{\beta + c^-}\right)\left(\frac{c^- - \beta}{c^- + \beta}\right)^{1/2} \mathcal{G}(ic^-)\, e^{-ic^- l} + O(l^{-3/2}), \quad \ldots(4.11)$$

and

$$\mathcal{G}(i\beta) = -B_\xi \left\{\frac{(K_1+\beta)(c^+ + \beta)}{(\beta - c^+)}\right\}^{1/2} \times \left\{\frac{(c^- - c^+)}{(K_1 + c^-)(K_1 + c^+)(\beta + c^+)}\right\} \times \exp\{i(c^+ l + K_1 l - \pi/4)\} - i\left(\frac{c^- - c^+}{\beta - c^+}\right)\left(\frac{c^+ + \beta}{c^+ - \beta}\right)^{1/2} \times \mathcal{I}(-ic^+)\, e^{ic^+ l} + O(l^{-3/2}). \quad \ldots(4.12)$$

It is to be noted that the terms of order $l^{-3/2}$ can be neglected in (4.11) and (4.12) for the evaluation of $\mathcal{I}(s)$ and $\mathcal{G}(s)$ up to the terms of $O(l^{-3/2})$.

Substituting these values from (4.11) and (4.12) in (4.9) and (4.10), we obtain the expressions for $\mathcal{I}(s)$ and $\mathcal{G}(s)$ in terms of $\mathcal{I}(-ic^+)$ and $\mathcal{G}(ic^-)$. Again, substituting $s = -ic^+$ and $s = ic^-$ in $\mathcal{I}(s)$ and $\mathcal{G}(s)$ respectively and solving the resulting equations for $\mathcal{I}(-ic^+)$ and $\mathcal{G}(ic^-)$, we obtain, upto $O(l^{-3/2})$:

$$\left.\begin{aligned} \mathcal{I}(-ic^+) &= \frac{1}{(1 - C_1C_0)}\left[(C_0A_1 + A_0) + \left\{(C_0B_1 + B_0 - A_0D_1) + \frac{C_0A_1 + A_0}{1 - C_1C_0}(D_0 + D_1)\right\}\frac{1}{l^{3/2}}\right] \\ &\text{and} \\ \mathcal{G}(ic^-) &= \frac{1}{(1 - C_1C_0)}\left[(C_1A_0 + A_1) + \left\{(C_1B_0 + B_1 - A_1D_0) + \frac{C_1A_0 + A_1}{1 - C_1C_0}(D_0 + D_1)\right\}\frac{1}{l^{3/2}}\right], \end{aligned}\right\} \quad \ldots(4.13)$$

where

$$
\left.
\begin{aligned}
A_0 &= B_\xi \cdot \frac{\{(K_1-\beta)(c^- - \beta)\}^{1/2}\, e^{(i\pi/4)-ic^-l}}{(\beta+c^-)^{1/2}(K_1+c^-)(K_1-c^+)}\\
A_1 &= -\,B_\xi \cdot \frac{\{(K_1+\beta)(c^+ + \beta)\}^{1/2}\, e^{ic^+l K_1 l-(i\pi/4)}}{(\beta-c^+)^{1/2}(K_1+c^-)(K_1+c^+)}\\
B_0 &= B_\xi \cdot \frac{(2\beta)^{-1/2}\, e^{-i\beta l}}{2\pi^{1/2}}\left[\frac{-(K_1-\beta)^{1/2}}{(K_1+\beta)(K_1+c^+)(\beta-c^-)}\right.\\
&\quad \left. - \frac{(c^- - c^+)\{(K_1+\beta)(c^+ + \beta)\}^{1/2}\, e^{i(c^+ + K_1)l}}{(\beta-c^-)(K_1+c^-)(K_1+c^+)(\beta-c^+)^{3/2}}\right]\\
B_1 &= B_\xi \cdot \frac{(2\beta)^{-1/2}\, e^{-i\beta l}}{2\pi^{1/2}}\left[\frac{-i(K_1+\beta)^{1/2}\, e^{iK_1 l}}{(K_1+c^-)(\beta+c^+)(\beta-K_1)}\right.\\
&\quad \left. + \frac{i(c^- - c^+)\{(K_1-\beta)(c^- - \beta)\}^{1/2}\, e^{-ic^-l}}{(\beta+c^+)(K_1+c^-)(K_1+c^+)(\beta+c^-)^{3/2}}\right]\\
C_0 &= -\,i\left(\frac{c^- - \beta}{c^- + \beta}\right)^{1/2} e^{-ic^-l}, \quad C_1 = -\,i\left(\frac{c^+ + \beta}{c^+ - \beta}\right)^{1/2} e^{ic^+l}\\
D_0 &= \frac{(2\beta)^{-1/2}}{2\pi^{1/2}} \cdot \frac{(c^- - c^+)(\beta+c^+)^{1/2}}{(\beta-c^-)(\beta-c^+)(c^+ - \beta)^{1/2}}\, e^{i(c^+ - \beta)l-(i\pi/4)}\\
D_1 &= \frac{(2\beta)^{-1/2}}{2\pi^{1/2}} \cdot \frac{(c^- - c^+)(c^- - \beta)^{1/2}}{(\beta+c^+)(\beta+c^-)^{3/2}}\, e^{-i(c^- + \beta)l-(i\pi/4)}.
\end{aligned}
\right\}
\quad \ldots(4.14)
$$

Substituting the values of $\mathcal{J}(i\beta)$ and $\mathcal{G}(i\beta)$ from (4.11) and (4.12) in (4.10) and (4.9), we may finally express $\mathcal{J}(s)$ and $\mathcal{G}(s)$ as :

$$
\left.
\begin{aligned}
\mathcal{J}(s) &= \frac{\lambda_1}{s+ic^-} + \frac{\lambda_2}{s+i\beta}\\
\text{and}\qquad\qquad &\\
\mathcal{G}(s) &= \frac{\lambda_3}{s-ic^+} + \frac{\lambda_4}{s+i\beta}
\end{aligned}
\right\}, \quad \ldots(4.15)
$$

where the constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ in (4.15) are given by :

$$
\lambda_1 = (c^- - c^+)\left(\frac{c^- - \beta}{c^- + \beta}\right)^{1/2} e^{-ic^-l}\left\{\mathcal{J}(ic^-) - B_\xi \cdot \frac{(K_1-\beta)^{1/2}\, e^{-i\pi/4}}{(K_1+c^-)(K_1+c^+)}\right\} \quad \ldots(4.16)
$$

$$
\begin{aligned}
\lambda_2 = \frac{(2\beta)^{-1/2}\, e^{-i\beta l}}{2\pi^{1/2}\, l^{3/2}}\Bigg[&-\frac{iB_\xi(K_1-\beta)^{1/2}}{(K_1+c^+)} \cdot \frac{(\beta-c^+)}{(\beta-c^-)(K_1+\beta)}\\
&- iB_\xi\left(\frac{c^- - c^+}{\beta-c^-}\right)\\
&\times \frac{\{(K_1+\beta)(c^+ + \beta)\}^{1/2}\, e^{i(c^+ + K_1)l}}{(K_1+c^-)(K_1+c^+)(\beta-c^+)^{1/2}}\\
&+ \frac{(c^- - c^+)(c^+ + \beta)^{1/2}}{(\beta-c^-)(c^+ - \beta)^{1/2}}\, \mathcal{J}(-ic^+)\, e^{ic^+l+(i\pi/4)}\Bigg]
\end{aligned}
\quad \ldots(4.17)
$$

$$\lambda_3 = (c^- - c^+)\, e^{ic^+l} \left[\left(\frac{\beta + c^+}{c^+ - \beta} \right)^{1/2} \mathcal{G}(-\, ic^+) \right.$$

$$\left. - B_\xi \cdot \frac{\{(K_1 + \beta)(c^+ + \beta)\}^{1/2}\, e^{iK_1 l + (i\pi/4)}}{(\beta - c^+)^{1/2}(K_1 + c^-)(K_1 + c^+)} \right] \qquad \ldots(4.18)$$

and

$$\lambda_4 = \frac{(2\beta)^{-1/2}\, e^{-i\beta l}}{2\pi^{1/2}\, l^{3/2}} \left[\frac{B_\xi e^{iK_1 l}(K_1 + \beta)^{1/2}(\beta + c^-)}{(K_1 + c^-)(\beta + c^+)(\beta - K_1)} - B_\xi \left(\frac{c^- - c^+}{\beta + c^+} \right) \right.$$

$$\times \frac{\{(K_1 - \beta)(c^- - \beta)\}^{1/2}\, e^{-ic^-l}}{(K_1 + c^-)(K_1 + c^+)(\beta + c^-)^{1/2}}$$

$$\left. + \frac{(c^- - c^+)(c^- - \beta)^{1/2}}{(\beta + c^+)(c^- + \beta)^{1/2}}\, \mathcal{G}(ic^-)\, e^{-ic^-l + (i\pi/4)} \right] \qquad \ldots(4.19)$$

where $\mathcal{G}(-\, ic^+)$ and $\mathcal{G}(ic^-)$ are given by (4.13) and (4.14).

If we now go back to (4.2), then with the help of (4.15), we obtain $F^*(s)$ and $G^*(s)$ and finally, by (3.22), we get :

$$P(s)\, A(s) = \frac{1}{(s + i\beta)^{1/2}} \left[\lambda_3 + \lambda_4 \left(\frac{s - ic^+}{s + i\beta} \right) - A_1' \left(\frac{s - ic^+}{s + iK_1} \right) \right]$$

$$+ \frac{e^{sl}}{i(s - i\beta)^{1/2}} \left[\lambda_1 + \lambda_2 \left(\frac{s - ic^-}{s - i\beta} \right) + A_2' \left(\frac{s - ic^-}{s + iK_1} \right) \right] \qquad \ldots(4.20)$$

where

$$\left. \begin{aligned} A_1' &= B_\xi \cdot \frac{(K_1 - \beta)^{1/2}}{(K_1 + c^+)}\, e^{i\pi/4}, \\ A_2' &= B_\xi \cdot \frac{(K_1 + \beta)^{1/2}}{(K_1 + c^-)}\, e^{iK_1 l - (i\pi/4)} \end{aligned} \right\} \qquad \ldots(4.21)$$

and where λ_1, λ_2, λ_3, λ_4 are given by (4.16) – (4.19).

This completes the determination of the unknown function $A(s)$, by means of which we can determine the field components with the help of (3.5), the Mellin's inversion formula and the formulae (2.13) – (2.14).

5. Evaluation of the Far-field

By (2.12), (3.5), (4.21) and Mellin's inversion formula, we obtain

$$E_\xi^{(s)}(x, y, z) = -\, e^{-iK_2 y} \cdot \frac{1}{2\pi i} \int_{\delta_1 - i\infty}^{\delta_1 + i\infty} \left[\frac{1}{(s + i\beta)^{1/2}} \left\{ \lambda_3 + \lambda_4 \left(\frac{s - ic^+}{s + i\beta} \right) - \right. \right.$$

(*equation continued on p.* 716)

$$- A_1'\left(\frac{s - ic^+}{s + iK_1}\right)\Big\} + \frac{e^{st}}{i(s - i\beta)^{1/2}}\left\{\lambda_1 + \lambda_2\left(\frac{s - ic^-}{s - i\beta}\right)\right.$$

$$\left. + A_2'\left(\frac{s - ic^-}{s + iK_1}\right)\right] \times e^{-i\kappa|z| + sx}\, ds, \qquad \ldots(5.1)$$

$$[\mathrm{Re}\,(-i\beta) < \delta_1 < \mathrm{Re}\,(i\beta)].$$

The other scattered field components can be derived in a similar manner. In this section, we shall concentrate only on the electric field component $E_\xi^{(s)}$.

The integral on the right of (5.1) can be evaluated approximately for $\beta r \gg 1$, where $x = r\cos\phi$, $|z| = r\sin\phi$, by the method of steepest descent, after taking care of the poles of the integrand (see Jones 1964), while deforming the contour. Note that the saddle point occurs at the point $s = -i\beta\cos\phi$, and the poles disappear, because of the relations (4.21).

We obtain, for $|\beta r| \gg 1$,

$$E_\xi^{(s)}(x, y, z) \approx \tilde{E}_\xi^{(s)}(x, y, z) \qquad \ldots(5.2)$$

where

$$\tilde{E}_\xi^{(s)} = -\left(\frac{\beta}{2\pi r}\sin^2\phi\right)^{1/2} \exp\left\{- iK_2 y - i\beta r + i\frac{\pi}{4}\right\}$$

$$\times\left[\frac{1}{(2i\beta)^{1/2}\sin\frac{1}{2}\phi}\left\{\lambda_3 - \lambda_4\left(\frac{c^+ + \beta\cos\phi}{2\beta\sin^2\frac{\phi}{2}}\right)\right.\right.$$

$$\left. + A_1'\left(\frac{c^+ + \beta\cos\phi}{K_1 - \beta\cos\phi}\right)\right\} + \frac{e^{-(i\beta l\cos\phi)}}{(2i\beta)^{1/2}\cos\frac{1}{2}\phi}$$

$$\left.\times\left\{\lambda_1 + \lambda_2\left(\frac{c^- + \beta\cos\phi}{2\beta\cos^2\frac{\phi}{2}}\right) - A_2'\left(\frac{c^- + \beta\cos\phi}{K_1 - \beta\cos\phi}\right)\right\}\right]. \qquad \ldots(5.3)$$

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