A NEW APPROACH TO STUDY M-SOLUTIONS OF THE LANE-EMDEN EQUATION OF INDEX 5

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A geometric technique for obtaining M-solutions of the Lane-Emden equation of index 5 for a non-rotating gas sphere has been developed with the hope that it will be more useful than the cumbersome process of numerical integrations. The solutions have been compared with the known results of numerical integrations. Astrophysical implications of our findings have been pointed out.

INTRODUCTION

The Lane-Emden equation of index 5, for a non-rotating gas sphere, governing the distribution of pressure and density, can be put in the form (Chandrasekhar 1939)

$$\frac{dz}{dt} = \pm \left[2D + \frac{z^2}{4} - \frac{z^6}{12} \right]^{1/2}. \qquad \text{...(1)}$$

Srivastava (1966), Srivastava and Sharma (1968a) and Sharma (1970) have shown that all classes of physically significant M-solutions, as derived from homologous transformations $\theta(\xi) \rightarrow A^{1/2}\,\theta(A\xi)$, are confined to the inequality

$$144D^2 - 1 < 0. \qquad \text{...(2)}$$

Srivastava and Sharma (1968b) have further given a series solution of (1) from which all classes of E-, M- and F-solutions can be derived. We feel that as regards numerical solutions of (1), for a few selected values of D, one has to tabulate a large number of values of z and $\pm\, dz/dt$, step by step at small intervals $(\pm \triangle t)$ and then obtain the corresponding values of θ and $-\,d\theta/d\xi$ in order to draw some astrophysical conclusions. The present method avoids this lengthy process.

The relations connecting z and t with ξ and θ are

$$z^2 = 2\theta^2\xi;\ \xi = e^{-t}. \qquad \text{...(3)}$$

For M-solutions, as ξ tends to zero, θ tends to infinity, but for other values of ξ, θ is finite. $t = 0$ corresponds to $\xi = 1$, at which θ is finite for all classes of solutions. In view of the homology theorem, there is clearly no loss of generality if at $t = 0$, we take $z = 1$. Hence, with the initial conditions

$$z = 1; \ \frac{dz}{dt} = \pm (2D + \tfrac{1}{6})^{1/2} \text{ at } t = 0 \qquad ...(4)$$

we seek a solution of (1) by graphical method for our chosen value of $D = -0.01$. This gives

$$\theta_0 = 0.7073; \ -\theta'_0 = 0.6243 \text{ at } \xi = 1 \qquad ...(5)$$

$'$ means differentiation of θ with respect to ξ. Following Piskunov (1974), the magnitude of the radius of curvature R of an integral curve, corresponding to the Lane-Emden equation of index 5 in (ξ, θ)-plane, is given by

$$| R | = \frac{1}{\cos^3 \phi((2/\xi) \tan\phi + \theta^5)}, \qquad ...(6)$$

where $d\theta/d\xi = \tan \phi$; ϕ being the angle between the positive axis and the tangent to the curve. From the foregoing, there follows a method of approximate construction of an integral curve by means of a smooth curve composed of arcs of circles. The arrangement of the M-solutions has been depicted in Fig. 1.

The Method of Constructing the Integral Curve: The Solution for $D = -0.01$

Through the point $M_0(\xi_0, \theta_0)$ draw a ray M_0T_0 with slope $\theta' = \tan \phi_0$ ($\phi_0 = 148^0\,02'$). From eqn. (6), we find the value $| R | = | R_0 | = 3.026$. Lay off a segment M_0C_0 (on a chosen scale) equal to $| R_0 |$ perpendicular to M_0T_0, and from the point C_0 (as centre) strike an arc M_0M_1 with radius $| R_0 |$. Let (ξ_1, θ_1) be the coordinates of the point M_1, which lies on the constructed arc and is sufficiently close to the point M_0, while $\tan \phi_1 (\phi_1 = 166^0)$ is the slope of the tangent M_1T_1 to the circle drawn at M_1. From eqn. (6), we find the value $| R | = | R_1 |$ that corresponds to M_1. Draw the segment $M_1C_1 = | R_1 | = 8.156$ (on a chosen scale) perpendicular to M_1T_1, and from C_1 (as centre) strike an arc M_1M_2 with radius $| R_1 |$. Then, on this arc take a point $M_2(\xi_2, \theta_2)$ close to M_1 and continue the construction as before until a sufficiently large piece of the curve consisting of the arcs of circles is obtained.

From the foregoing, it is clear that this curve is approximately an integral curve that passes through the point M_0. Obviously, the smallar the arcs M_0M_1, M_1M_2, M_2M_3, ..., the closer will the constructed curve be to the integral curve. Values of some of the physical quantities, such as, ξ and $d\theta/d\xi$ (or θ), which can be read off directly from Fig. 1, are found to be in good agreement with some of earlier results of the author (Table 1) obtained by the method of numerical integrations.

Graphical solutions for other values of $D = -0.02, -0.03, -0.04, -0.05, -0.06, -0.07, -0.08, -0.09, -0.10$ and -0.11, can be given in a similar way.

From the preceding discussions, we may conclude that the present geometrical approach for obtaining M-solutions of the Lane-Emden equation of index 5 is easier

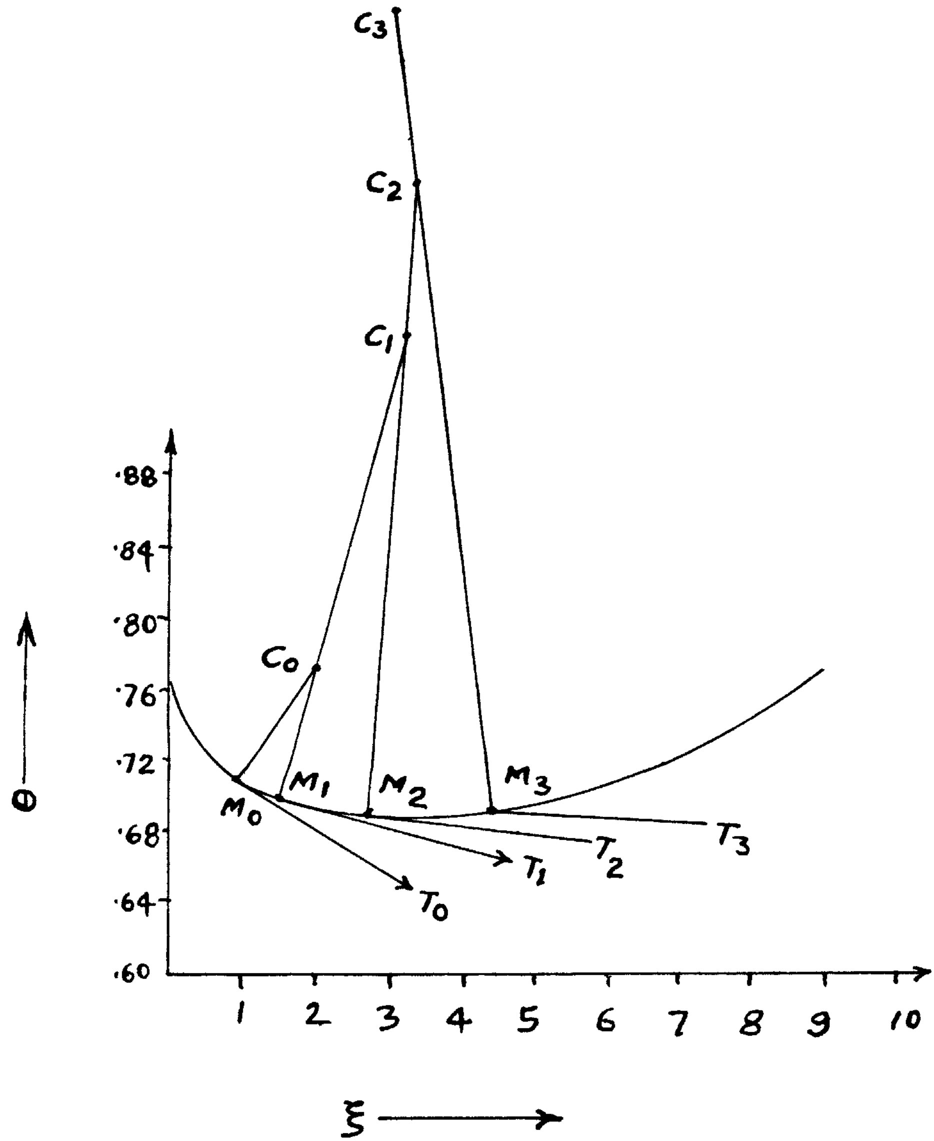

FIG. 1. M-solution of the Lane-Emden equation of index 5 for $D = -0.01$.

and more useful than the well-known methods of numerical integrations (Kunz 1957), in which, as mentioned above, we have to first compute the values of z and $\pm\, dz/dt$ to obtain the desired values of $d\theta/d\xi$ (or θ) and ξ. In the present method, these values are obtained directly.

ASTROPHYSICAL IMPLICATIONS

Values of some useful physical quantities, such as θ, $d\theta/d\xi$ and ξ which help in describing the physical structure of a star, can be read off directly from Fig. 1. The

TABLE I

*The solution**

$(D = -0.01)$

t	z	$\pm dz/dt$	ξ	$-d\theta/d\xi$
0.50	1.915	0.310	0.6065	1.346†
0.00	1.0000	.383	1.0000	0.624
− 0.50	0.8085	.347	1.6480	0.251
− 1.00	0.6350	.274	2.7183	0.093
− 1.50	0.4980	.202	4.4820	0.034
...	...	...	...	...

*The choice of convexity downwards (or upwards) of the arcs of circles is important, so as to satisfy the physical needs of the problem.

†Values of ξ and $d\theta/d\xi$ (or θ) appearing in the first row are not included in the discussion, looking to the validity of the physical structure of the star.

points M_0, M_1, M_2, M_3, ..., lying on arcs of the circles M_0M_1, M_1M_2, M_2M_3, ..., would give values of ξ (reduced radius) and $d\theta/d\xi$ (or θ). From a knowledge of these variables, many other astrophysical consequences could be drawn. For example, the values of two homology invariants u and v defined by $u = -\frac{\xi\theta^5}{\theta'} = 3 \times$ local density $\rho(r)$/mean density $\bar{\rho}(r)$ within radius r; and $v = -\frac{\xi\theta'}{\theta} = \frac{1}{(n+1)(\gamma-1)}$ gravitational energy/internal energy, can be calculated.

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