AN EDGE CRACK IN A THIN SEMI-CIRCULAR PLATE

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We consider under the assumptions of plane strain, the problem of determining stress and displacement fields in an elastic thin semi-circular plate containing an edge crack normal to the diameter of the plate. The crack faces are subjected to a general distribution of pressure. The theory of dual integral equations is used for reducing the mixed boundary value problem to that of solving a pair of simultaneous equations of Fredholm type. These equations are solved numerically. The expressions for the stress intensity factor and the crack energy are derived.

1. Introduction

A problem of great importance in fracture mechanics is to consider a thin sheet in the shape of a semi-circle containing a line crack perpendicular to the diameter of the plate and loaded by internal pressure. The problem of an edge crack in an elastic half plane has been considered by Koiter (1956, 1965), Wigglesworth (1957), Doran and Buchwald (1969), Stallybrass (1970) and Sneddon and Das (1971). Among these, Stallybrass (1970) is the first author who has considered the problem for a general pressure distribution of the type $p(x) = x^{a-1}$. His method is based on the solution of an integral equation for a very convenient auxiliary variable. The integral equation is solved by the Wiener-Hopf technique. Sneddon and Das use the theory of integral transform and dual integral equations to solve the same problem for a very general pressure distribution $p(x)$ with the assumption that the pressure distribution is same on opposite faces of the crack so that only quarter plane problem need be considered.

In the present paper we shall study the problem of determining stress and displacement fields in the neighbourhood of an edge crack in a thin semi-circular plate of radius c. The crack is subjected to an internal pressure distribution $p(r)$ which is same on the opposite faces. The corresponding mixed boundary value problem is solved by suitably representing the complex potentials and using the theory of dual integral equations. In section 5 we show how the resulting pair of simultaneous Fredholm integral equations are solved numerically by reducing them to a set of linear algebraic simultaneous equations. These equations are solved for the case in which the applied internal pressure is constant. In section 6, we have found quantities of physical interest like the stress intensity factor and the crack energy.

2. Basic Equations of Two-Dimensional Elasticity

We quote from England (1971, p. 44) the basic equations of two-dimensional elasticity. In plane polar coordinates (r, θ), we have

$$\mu (u_r + iu_\theta) = [k\Omega (z) - z\overline{\Omega' (z)} - \overline{W' (z)}]\, e^{-i\theta} \qquad ...(2.1)$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 4 [\Omega' (z) + \overline{\Omega' (z)}] \qquad ...(2.2)$$

$$\sigma_{rr} + i\sigma_{r\theta} = 2 [\Omega' (z) + \overline{\Omega' (z)}] - 2 [\overline{z\Omega'' (z)} + \overline{W'' (z)}]\, e^{-2i\theta} \qquad ...(2.13)$$

$$\sigma_{\theta\theta} - i\sigma_{r\theta} = 2 [\Omega' (z) + \overline{\Omega' (z)}] + 2 [z\overline{\Omega'' (z)} + \overline{W'' (z)}]\, e^{-2i\theta} \qquad ...(2.4)$$

where (u_r, u_θ) and $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})$ are the components of displacements and stresses respectively. μ and η respectively represent rigidity modulus and Poisson's ratio of the material of the plate; $k = (3 - 4\eta)$ for plane strain and $k = (3 - \eta)/(1 + \eta)$ for generalized plane stress. For convenience, let

$$\Omega (z) = \Omega_1 (z) + \Omega_2 (z) \qquad ...(2.5)$$

$$W' (z) = - z\Omega_1' (z) + \Omega_1 (z) + z\Omega_2' (z) - \Omega_2 (z) + \chi (z) \qquad ...(2.6)$$

such that

$$W'' (z) = - z\Omega_1'' (z) + z\Omega_2'' (z) + \chi' (z).$$

3. Formulation of the Problem

We consider a thin semi-circular plate of homogeneous and isotropic material. The plate contains a crack perpendicular to the diameter and the crack is opened by internal normal pressure $p (r)$. The boundaries of the plate are supposed to be stress free. Hence by symmetry of the problem, we need only consider quarter plate problem. Thus the plate is supposed to occupy the region $0 \leqslant r \leqslant c$ $(c > 1)$, $0 \leqslant \theta \leqslant \pi/2$ and the crack occupies the region $0 \leqslant r \leqslant 1$, $\theta = 0$ in the plate. For symmetric deformation the displacement vector U is supposed to take values $(u_r, u_\theta, 0)$ and non-vanishing components of stress tensor are σ_{rr}, $\sigma_{\theta\theta}$, and $\sigma_{r\theta}$. The boundary conditions on $\theta = 0$ may be written as

$$\left.\begin{array}{lll} (a) & \sigma_{\theta\theta} (r, 0) = - p (r), & 0 \leqslant r \leqslant 1 \\ (b) & \sigma_{r\theta} (r, 0) = 0, & 0 \leqslant r \leqslant c \\ (c) & u_\theta (r, 0) = 0, & 1 \leqslant r \leqslant c. \end{array}\right\} \qquad ...(3.1)$$

Since the plate is assumed to be stress free we have, in addition to the above, on the free boundary

$$\sigma_{r\theta} (c, \theta) = \sigma_{rr} (c, \theta) = 0, \quad 0 \leqslant \theta \leqslant \theta/2 \qquad ...(3.2)$$

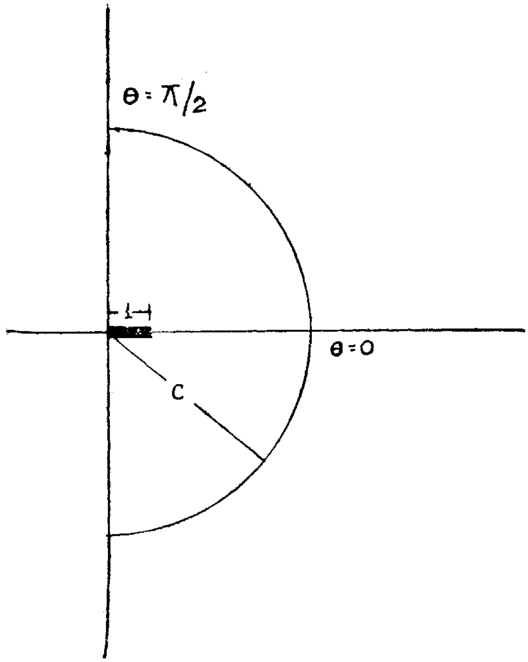

FIG. 1

and on $\theta = \pi/2$

$$\sigma_{\theta\theta}\ (r, \pi/2) = \sigma_{r\theta}\ (r, \pi/2) = 0, \qquad r \geqslant 0. \qquad \ldots(3.3)$$

For the problem under consideration, the appropriate expressions for the complex potentials are

$$\Omega_1(z) = \frac{i}{4} \int_0^\infty \xi^{-1} B(\xi)\, e^{-i\xi z}\, d\xi + \sum_{n=0}^{\infty} \frac{p_n\, z^{n+1}}{n+1} \qquad \ldots(3.4)$$

$$\Omega_2(z) = \tfrac{1}{4} \int_0^\infty \xi^{-1} A(\xi)\, e^{-i\xi z}\, d\xi \qquad \ldots(3.5)$$

and

$$\chi(z) = \sum_{n=0}^{\infty} \frac{q_n z^{n+1}}{n+1}\,. \qquad \ldots(3.6)$$

Substituting these and the other appropriate quantities in equations (2.1) to (2.4) and separating the real and imaginary parts of each equation we obtain that

$$\mu u_r(r, \theta) = \int_0^\infty \xi^{-1} B(\xi)\, e^{-\xi r \sin\theta}\, [(2 - 2\eta + \xi r \sin\theta) \sin\theta$$

$$\times \cos(\xi r \cos\theta) - (1 - 2\eta - \xi r \sin\theta) \cos\theta \sin(\xi r \cos\theta)]\, d\xi +$$

[equation continued on page 75]

$$+ \int_0^\infty \xi^{-1} A(\xi) e^{-\xi r \cos\theta} [(2 - 2\eta + \xi r \cos\theta) \cos\theta \cos(\xi r \sin\theta)$$

$$+ (1 - 2\eta - \xi r \cos\theta) \sin\theta \sin(\xi r \sin\theta)] \, d\xi$$

$$- \sum_{n=0}^\infty [n a_n r^{n-1} + (n - 4\eta - 2) b_n r^{n+1}] \cos n\theta \qquad \ldots(3.7)$$

$$\mu u_\theta(r, \theta) = \int_0^\infty \xi^{-1} B(\xi) e^{-\xi r \sin\theta} [(2 - 2\eta + \xi r \sin\theta) \cos\theta$$

$$\times \cos(\xi r \cos\theta) + (1 - 2\eta - \xi r \sin\theta) \sin\theta \sin(\xi r \cos\theta)] \, d\xi$$

$$+ \int_0^\infty \xi^{-1} A(\xi) e^{-\xi r \cos\theta} [(1 - 2\eta - \xi r \cos\theta) \cos\theta \sin(\xi r \sin\theta)$$

$$- (2 - 2\eta + \xi r \cos\theta) \sin\theta \cos(\xi r \sin\theta)] \, d\xi$$

$$+ \sum_{n=0}^\infty [n a_n r^{n-1} + b_n (n - 4\eta + 4) r^{n+1}] \sin n\theta \qquad \ldots(3.8)$$

$$\sigma_{\theta\theta}(r, \theta) = \int_0^\infty A(\xi) e^{-\xi r \cos\theta} [\xi r \cos\theta \cos(2\theta - \xi r \sin\theta)$$

$$- \cos(\xi r \sin\theta)] \, d\xi - \int_0^\infty B(\xi) e^{-\xi r \sin\theta} [\xi r \sin\theta \cos(2\theta$$

$$+ \xi r \cos\theta) + \cos(\xi r \cos\theta)] \, d\xi + \sum_{n=0}^\infty [n(n-1) a_n r^{n-2}$$

$$+ (n+1)(n+2) b_n r^n] \cos n\theta \qquad \ldots(3.9)$$

$$\sigma_{r\theta}(r, \theta) = \int_0^\infty A(\xi) e^{-\xi r \cos\theta} [\xi r \cos\theta \sin(2\theta - \xi r \sin\theta)] \, d\xi$$

$$- \int_0^\infty B(\xi) e^{-\xi r \sin\theta} [\xi r \sin\theta \sin(2\theta + \xi r \cos\theta)] \, d\xi$$

$$+ \sum_{n=0}^\infty [n(n-1) a_n r^{n-2} + n(n+1) b_n r^n] \sin n\theta \qquad \ldots(3.10)$$

$$\sigma_{rr}(r, \theta) = - \int_0^\infty A(\xi) e^{-\xi r \cos\theta} [\xi r \cos\theta \cos(2\theta - \xi r \sin\theta)$$

$$+ \cos(\xi r \sin\theta)] \, d\xi - \int_0^\infty B(\xi) e^{-\xi r \sin\theta} [\cos(\xi r \cos\theta)$$

$$- \xi r \sin\theta \cos(2\theta + \xi r \cos\theta)] \, d\xi - \sum_{n=0}^\infty [n(n-1) a_n r^{n-2}$$

$$+ (n+1)(n-2) b_n r^n] \cos n\theta \qquad \ldots(3.11)$$

where

$$n(n-1)\,a_n = q_{n-2} - (n-2)\,p_{n-2}$$

$$(n+1)\,b_n = p_n. \qquad \ldots(3.12)$$

4. Reduction to Simultaneous Dual Integral Equations

From (3.10) we see that the boundary condition (3.1b) is automatically satisfied if all the odd coefficients a_{2n+1} and b_{2n+1} are zero. The boundary conditions (3.1a) and (3.1c) lead to the dual integral equations.

$$\frac{d}{dr}\left[\int_0^\infty \xi^{-1} B(\xi)\sin \xi r\, d\xi + r\int_0^\infty A(\xi)\, e^{-\xi r}\, d\xi\right] = F(r), \qquad 0 \leqslant r \leqslant 1 \qquad \ldots(4.1)$$

$$\int_0^\infty \xi^{-1} B(\xi) \cos \xi r\, d\xi = 0, \qquad 1 \leqslant r \leqslant c \qquad \ldots(4.2)$$

where

$$F(r) = p(r) + \sum_{n=0}^{\infty} (2n+1)(2n+2)(a_{2n+2} + b_{2n})\, r^{2n}.$$

We further see that one of the boundary conditions (3.3) is automatically satisfied while the other leads to the equation

$$\frac{d}{dr}\left[\int_0^\infty \xi^{-1} A(\xi)\sin \xi r\, d\xi + r\int_0^\infty B(\xi)\, e^{-\xi r}\, d\xi\right] = G(r), \qquad r \geqslant 0 \qquad \ldots(4.3)$$

where

$$G(r) = \sum_{n=0}^{\infty} (-1)^n (2n+1)(2n+2)(a_{2n+2} + b_{2n})\, r^{2n}.$$

The dual integral equations (4.1) and (4.2) admit solution of the type

$$B(\xi) = \xi \int_0^1 t\, g(t)\, J_0(\xi t)\, dt. \qquad \ldots(4.4)$$

If we take a similar solution for $A(\xi)$

$$A(\xi) = \xi \int_0^c t\, h(t)\, J_0(\xi t)\, dt, \qquad \ldots(4.5)$$

we find that equations (4.1) and (4.3) reduce to

$$\frac{d}{dr}\int_0^r \frac{t\, g(t)\, dt}{(r^2 - t^2)^{\frac{1}{2}}} + \frac{d}{dr}\left\{ r^2 \int_0^c \frac{t\, h(t)\, dt}{(t^2 + r^2)^{3/2}}\right\} = F(r), \qquad 0 \leqslant r \leqslant 1$$

$$\int_0^r \frac{t\,h(t^2)\,dt}{(r^2-t^2)^{\frac{1}{2}}} + \frac{d}{dr}\left\{r^2 \int_0^1 \frac{t\,g(t)\,dt}{(t^2+r^2)^{3/2}}\right\} = G(r), \qquad r \geqslant 0.$$

Inverting these Abel equations we get

$$g(t) + \int_0^c K(t,u)\,h(u)\,du = \phi(t), \qquad 0 \leqslant t \leqslant 1 \qquad \ldots(4.6)$$

$$h(t) + \int_0^1 K(t,u)\,g(u)\,du = \psi(t), \qquad t \geqslant 0 \qquad \ldots(4.7)$$

where

$$K(t,u) = \frac{4tu^2}{\pi(u^2+t^2)^2} \qquad \ldots(4.8)$$

$$\phi(t) = \frac{2}{\pi}\int_0^t \frac{F(r)\,dr}{(t^2-r^2)^{\frac{1}{2}}} \qquad \ldots(4.9)$$

$$\psi(t) = \frac{2}{\pi}\int_0^t \frac{G(r)\,dr}{(t^2-r^2)^{\frac{1}{2}}}\,. \qquad \ldots(4.10)$$

Thus the solution of the mixed boundary value problem reduces to the solution of the set of simultaneous Fredholm integral equations (4.6) and (4.7).

5. Conditions on the Free Boundary

We are yet to apply boundary conditions (3.2). We complete the solution by applying these conditions. Substitute for $A(\xi)$ and $B(\xi)$ in (3.10) and (3.11), perform the change of order of integration and then use the results of the appendix for $r = c$ to obtain the following equations:

$$-\sum_{n=0}^{\infty}[2n(2n-1)\,a_{2n}\,c^{2n-2} + (2n+1)(2n-2)\,b_{2n}\,c^{2n}]\cos 2n\theta$$

$$= \left[\sum_{n=1}^{\infty}\frac{(\frac{1}{2})_n\,2n\,(2n+1)}{n!\,c^{2n}}(A_{n-1}-B_{n-1})\right.$$

$$\left. + \sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n\,(n+1)(2n+1)}{n!\,c^{2n+2}}(A_n+B_n)\right]\cos 2n\theta \qquad \ldots(5.1)$$

$$\sum_{n=0}^{\infty} [2n(2n-1) a_{2n} c^{2n-2} + 2n(2n+1) b_{2n} c^{2n}] \sin 2n\theta$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!\, c^{2n}} \Big[2n^2 (A_{n-1} - B_{n-1}) + \frac{(n+1)(2n+1)}{c^2} (A_n + B_n) \Big] \sin 2n\theta \qquad \ldots(5.2)$$

where

$$A_n = (-1)^n \int_0^c t^{2n+1} n(t)\, dt \qquad \ldots(5.3)$$

$$B_n = \int_0^1 t^{2n+1} g(t)\, dt \qquad \ldots(5.4)$$

and

$$(\tfrac{1}{2})_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}.$$

From (5.1) and (5.2), we find that

$$2b_0 = \frac{1}{c^2}(A_0 + B_0) \qquad \ldots(5.5)$$

and for $n \geqslant 1$

$$2n(2n-1) a_{2n} c^{2n-2} + (2n+1)(2n-2) b_{2n} c^{2n}$$

$$= -\frac{(\frac{1}{2})_n}{n!\, c^{2n}} \Big[2n(n+1)(A_{n-1} - B_{n-1}) + \frac{(2n+1)(n+1)}{c^2} (A_n + B_n) \Big]$$

$$2n(2n-1) a_{2n} c^{2n-2} + 2n(2n+1) b_{2n} c^{2n}$$

$$= \frac{(\frac{1}{2})_n}{n!\, c^{2n}} \Big[2n^2 (A_{n-1} - B_{n-1}) + \frac{(2n+1)(n+1)}{c^2} (A_n + B_n) \Big]$$

From these equations we derive that

$$a_{2n+2} + b_{2n} = \frac{(\frac{1}{2})_n}{n!\, c^{4n}} \Big[n(A_{n-1} - B_{n-1}) + \frac{2(n+1)}{c^2} B_n - \frac{(2n+3)(2n+1)(n+2)}{4(n+1)^2 c^4} (A_{n+1} + B_{n+1}) \Big]. \qquad \ldots(5.6)$$

Substituting the values of these coefficients in (4.9), (4.10) and re-arranging (4.6) and (4.7) we obtain

$$g(t) + \int_0^c [K(t, u) + K_1(t, u)]\, h(u)\, du$$

$$+ \int_0^1 K_2(t, u)\, g(u)\, du = \frac{2}{\pi} \int_0^t \frac{p(r)\, dr}{(t^2 - r^2)^{\frac{1}{2}}}, \quad 0 \leqslant t \leqslant 1 \qquad \ldots(5.7)$$

and

$$h(t) + \int_0^1 [K(t, u) + L_1(t, u)]\, g(u)\, du + \int_0^c L_2(t, u)\, h(u)\, du = 0, \quad t \geqslant 1 \qquad \ldots(5.8)$$

where

$$K_1(t, u) = \frac{u}{c^2} - \frac{3u^3}{c^4} + K_1'(t, u) \qquad \ldots(5.9)$$

$$K_2(t, u) = -\frac{3u}{c^2} + \frac{3u^3}{c^4} + K_2'(t, u) \qquad \ldots(5.10)$$

$$L_1(t, u) = -\frac{3u}{c^2} + \frac{3u^3}{c^4} + K_1'(t, u) \qquad \ldots(5.11)$$

$$L_2(t, u) = \frac{u}{c^2} - \frac{3u^3}{c^4} + K_2'(t, u) \qquad \ldots(5\ 12)$$

with

$$K_1'(t, u) = \sum_{n=1}^{\infty} \frac{(-1)^n (\frac{1}{2})_n (\frac{1}{2})_n (2n+1)(2n+2)}{(n!)^2} \left(\frac{ut}{c^2}\right)^{2n}$$

$$\times \frac{1}{u}\left[n - \frac{(2n+3)(2n+1)(n+2)\, u^4}{4(n+1)^2 c^4}\right]$$

$$K_2'(t, u) = \sum_{n=1}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n (2n+1)(2n+2)}{(n!)^2} \left(\frac{ut}{c^2}\right)^{2n}$$

$$\times \frac{1}{u}\left[n + \frac{(2n+3)(2n+1)(n+2)}{4(n+1)^2 c^4}\, u^4 - \frac{2(n+1)}{c^2}\, u^2\right].$$

The simultaneous integral equations (5.7) and (5.8) are solved numerically for the physically important case in which the applied internal pressure $p(r)$ is constant and is equal to p_0. In this case, these integral equations can be written as

$$g_1(t) + c\int_0^1 [K(t, cz) + K_1(t, cz)]\, h_1(cz)\, dz + \int_0^1 K_2(t, u)\, g_1(u)\, du = 1 \qquad \ldots(5.13)$$

$$h_1(ct) + \int_0^1 [K(ct, u) + L_1(ct, u)] g_1(u)\, du + c \int_0^1 L_2(t, cz) h_1(cz)\, dz = 0 \quad \text{...(5.14)}$$

where

$$g_1(t) = \frac{g(t)}{p_0}, \qquad h_1(t) = \frac{h(t)}{p_0} \quad \text{...(5.15)}$$

and

$$z = u/c.$$

These integral equations are solved by the method of Fox and Goodwin (1953). The calculations are done for two values of the radius of semi-circular plate (*i.e.*, for $c = 2$ and 5). The numerical values of $g_1(t)$ and $h_1(ct)$ are given below.

t	$c = 2$		$c = 5$	
	$g_1(t)$	$h_1(ct)$	$g_1(t)$	$h_1(ct)$
0·0	— 3·280340	— 5·165807	— 1·003157	—1·795511
0·1	—124·270390	—96·652767	—20·497377	9·073520
0·2	— 4·913014	8·425945	— 7·692865	1·090898
0·3	—10·436752	12·557643	— 2·964657	0·206548
0·4	— 8·914321	— 0·799269	— 4·020847	0·349107
0·5	— 7·164318	4·230342	— 1·452954	0·484806
0·6	— 5·873777	2·028554	— 1·078147	1·564827
0·7	— 5·031958	0·048282	— 0·879214	1·893466
0·8	— 4·406386	— 0·159063	— 0·299509	1·876803
0.9	— 6·840463	— 0·056502	— 0·676790	—0·816932
1·0	— 3·465382	0·654422	— 1·026032	—0·650001

6. Quantities of Physical Interest

On putting $\theta = 0$ in equation (3.9), we get

$$\sigma_{\theta\theta}(r, 0) = -\int_0^\infty A(\xi)(1 - \xi r) e^{-\xi r}\, d\xi - \int_0^\infty B(\xi) \cos \xi r\, d\xi$$

$$+ \sum_{n=0}^{\infty} [2n(2n-1) a_{2n} r^{2n-2} + (2n+1)(2n+2) b_{2n} r^{2n}].$$

Substituting the values of $A(\xi)$ and $B(\xi)$ from (4.4) and (4.5) we find that

$$\sigma_{\theta\theta}(r, 0) = -\frac{d}{dr}\left[\int_0^1 \frac{t\,g(t)\,dt}{(r^2 - t^2)^{\frac{1}{2}}} + r^2 \int_0^c \frac{t\,h(t)\,dt}{(r^2 + t^2)^{3/2}}\right], \quad r > 1$$

$$= \frac{p_0\, r}{(r^2 - 1)^{\frac{1}{2}}}\, g_1(1) + 0(1), \quad r > 1. \qquad \ldots(6.1)$$

Now the stress intensity factor at the tip of the edge crack is given by

$$K = \lim_{r \to 1+} (r - 1)^{\frac{1}{2}}\, \sigma_{\theta\theta}(r, 0), \quad r > 1. \qquad \ldots(6.2)$$

Hence

$$K = \frac{p_0}{\sqrt{2}}\, g_1(1) + 0(1), \quad r > 1. \qquad \ldots(6.3)$$

The normal component of displacement of the crack surface is given by

$$2\mu u_\theta(r, 0) = 2(1 - \eta) \int_0^\infty \xi^{-1} B(\xi) \cos \xi r\, d\xi, \qquad 0 \leqslant r \leqslant 1.$$

Substituting for $B(\xi)$ from (4.4) we find that

$$u_\theta(r, 0) = \frac{2(1 - \eta^2)\, p_0}{E} \int_r^1 \frac{t\, g_1(t)\, dt}{(t^2 - r^2)^{\frac{1}{2}}}, \quad 0 \leqslant r \leqslant 1 \qquad \ldots(6.4)$$

where E is the Young's modulus for the material of the plate.

Further the work done in opening the crack is given by

$$W = \int_0^1 p(r)\, u_\theta(r, 0)\, dr$$

$$= p_0 \int_0^1 u_\theta(r, 0)\, dr. \qquad \ldots(6.5)$$

Substituting the value of $u_\theta(r, 0)$ from (6.4) we get

$$W = \frac{\pi(1 - \eta^2)\, p_0^2}{E} \int_0^1 t\, g_1(t)\, dt. \qquad \ldots(6.6)$$

The numerical values of stress intensity factor and crack energy are given below:

c	$\sqrt{2}K/p_0$	$EW/\pi(1 - \eta^2)\, p_0^2$
2	3·465382	13·9721900
5	1·026032	8·8671576

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APPENDIX

It is easy to derive following results:

$$\int_0^\infty e^{-\xi r \sin\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r\cos\theta) \\ \sin(\xi r \sin\theta) \end{Bmatrix} d\xi$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n\, t^{2n}}{n!\, r^{2n+1}} \begin{Bmatrix} \sin(2n+1)\theta \\ \cos(2n+1)\theta \end{Bmatrix}$$

$$\int_0^\infty e^{-\xi r \cos\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r\ \sin\theta) \\ \sin(\xi r\ \sin\ \theta) \end{Bmatrix} d\xi$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2})_n\, t^{2n}}{n!\ r^{2n+1}} \begin{Bmatrix} \cos(2n+1)\theta \\ \sin(2n+1)\theta \end{Bmatrix}$$

and therefore

$$\int_0^\infty \xi e^{-\xi r \sin\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r\cos\theta) \\ \sin(\xi r\ \cos\theta) \end{Bmatrix} d\xi$$

$$= \mp \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n\,(2n+1)\, t^{2n}}{n!\ r^{2n+2}} \begin{Bmatrix} \cos(2n+2)\theta \\ \sin(2n+2)\theta \end{Bmatrix}$$

$$\int_0^\infty \xi^2 e^{-\xi r \sin\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r \cos\theta) \\ \sin(\xi r \cos\theta) \end{Bmatrix} d\xi$$

$$= -\sum_{n=0}^{\infty} \frac{(\tfrac{1}{2})_n (2n+1)(2n+2)\, t^{2n}}{n!\, r^{2n+3}} \begin{Bmatrix} \sin(2n+3)\theta \\ \cos(2n+3)\theta \end{Bmatrix}$$

$$\int_0^\infty \xi e^{-\xi r \cos\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r \sin\theta) \\ \sin(\xi r \sin\theta) \end{Bmatrix} d\xi$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(\tfrac{1}{2})_n (2n+1)\, t^{2n}}{n!\, r^{2n+2}} \begin{Bmatrix} \cos(2n+2)\theta \\ \sin(2n+2)\theta \end{Bmatrix}$$

$$\int_0^\infty \xi^2 e^{-\xi r \cos\theta} J_0(\xi t) \begin{Bmatrix} \cos(\xi r \sin\theta) \\ \sin(\xi r \sin\theta) \end{Bmatrix} d\xi$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(\tfrac{1}{2})_n (2n+1)(2n+2)\, t^{2n}}{n!\, r^{2n+3}} \begin{Bmatrix} \cos(2n+3)\theta \\ \sin(2n+3)\theta \end{Bmatrix}.$$