A GRAVITATIONALLY NON-DEGENERATE VISCOUS FLUID COSMOLOGICAL MODEL IN GENERAL RELATIVITY

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A plane symmetric cosmological model representing a viscous fluid with free gravitational field of non-degenerate type 1 has been derived. The effect of viscosity on various kinematical parameters has been discussed.

1. Introduction

In recent years, astronomical observations have tended to confirm a degree of spatial anisotropy in the large scale behaviour of the universe. Homogeneous models with spatial anisotropy have been studied widely. Particular examples are the plane symmetric cosmological models in which the matter distribution is that of a perfect fluid. A cosmological model which is cylindrically symmetric and of non-degenerate Petrov type 1 has been derived by Roy and Singh (1976). In our earlier paper, we have derived some viscous fluid cosmological models of plane symmetry in which the free gravitational field is of degenerate Petrov type 1 (Roy and Prakash 1976). In this paper, we construct a gravitationally non-degenerate viscous fluid cosmological model of plane symmetry which represents a shearing and geodetic but non-rotating fluid flow. It is found that the kinematic viscosity prevents the shear, the expansion and the free gravitational field to wither away for large values of t.

2. Field Equations

We consider the plane-symmetric metric in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2 \qquad \ldots(1)$$

where A, B, C are functions of t alone. The energy-momentum tensor for a viscous fluid distribution is given by Landau and Lifschitz (1963)

$$T_i^j = (\epsilon + p)\, V_iV^j + pg_i^j - \eta(V_{i,}^j + V_{,i}^j + V^jV^\rho V_{i,\rho} + V_iV^\rho V_{,\rho}^j)$$

$$- (\zeta - \tfrac{2}{3}\eta)\, V_{,\rho}^\rho\, (g_i^j + V_iV^j) \qquad \ldots(2)$$

together with

$$g_{ij}V^iV^j = -1 \qquad \ldots(3)$$

p being the pressure, ϵ the density, η and ζ the two coefficients of viscosity and a comma signifies covariant differentiation. V^i is the unit velocity vector satisfying (3). We assume the co-ordinates to be comoving, so that

$$V^1 = V^2 = V^3 = 0 \text{ and } V^4 = \frac{1}{A}.$$

The field equations

$$R_i^j - \tfrac{1}{2} R \delta_i^j + \Lambda \delta_i^j = -8\pi T_i^j \quad \ldots(4)$$

for the line-element (1) are as follows :

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_4 C_4}{BC} + \frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC}\right] - \Lambda = 8\pi\left[p - 2\eta \frac{A_4}{A^2} - (\zeta - \tfrac{2}{3}\eta) V^\rho_{;\rho}\right] \quad \ldots(5)$$

$$\frac{1}{A^2}\left[-\frac{C_{44}}{C} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right] - \Lambda = 8\pi\left[p - 2\eta \frac{B_4}{AB} - (\zeta - \tfrac{2}{3}\eta) V^\rho_{;\rho}\right] \quad \ldots(6)$$

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right] - \Lambda = 8\pi\left[p - 2\eta \frac{C_4}{AC} - (\zeta - \tfrac{2}{3}\eta) V^\rho_{;\rho}\right] \quad \ldots(7)$$

$$\frac{1}{A^2}\left[\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{A_4 C_4}{AC}\right] + \Lambda = 8\pi\epsilon. \quad \ldots(8)$$

The suffix 4 after the symbols A, B, C denotes ordinary differentiation with respect to t. These are four equations in five unknowns, A, B, C, ϵ and p. The coefficients of viscosity η and ζ are taken as constants. Equations (5) – (8) are not independent, but they are related by the contracted Bianchi identities (Roy and Prakash 1976).

For complete solution of eqns. (5) – (8), we need an extra condition. Here, we assume that $C^{14}_{14} = C^{23}_{23} = 0$. The resulting space-time will obviously be of non-degenerate Petrov type 1. If we take any of the two metric potentials to be equal, the space-time becomes conformal to flat and the pressure and the density terms arise only due to viscosity, which does not correspond to a realistic distribution. We, therefore, consider A, B, C to be unequal. From eqns. (5) and (6), we obtain

$$\frac{A_{44}}{A} - \frac{A_4^2}{A^2} + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) - \frac{B_{44}}{B} - \frac{B_4 C_4}{BC} = 16\pi\eta A\left[\frac{B_4}{B} - \frac{A_4}{A}\right]. \quad \ldots(9)$$

From eqns. (6) and (7), we get

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} = 16\pi\eta A\left[\frac{C_4}{C} - \frac{B_4}{B}\right] \quad \ldots(10)$$

The condition $C_{14}^{14} = C_{23}^{23} = 0$ leads to

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} - 2\,\frac{A_{44}}{A} + 2\,\frac{A_4^2}{A^2} - 2\,\frac{B_4C_4}{BC} = 0. \qquad \text{...(11)}$$

Equations (9) and (10) lead to

$$A = LB\left(\frac{B}{C}\right)^K \qquad \text{...(12)}$$

where K and L are constants of integration. From eqns. (11) and (12), we obtain

$$K\left[\frac{B_4}{B} - \frac{C_4}{C}\right]_4 + \frac{B_4C_4}{BC} + \tfrac{1}{2}\left[\frac{B_{44}}{B} - \frac{C_{44}}{C}\right] - \frac{B_4^2}{B^2} = 0 \qquad \text{...(13)}$$

Also, from eqns. (10) and (12), we get

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} = -\,16\pi\eta LB\left[\frac{B}{C}\right]^K\left[\frac{B_4}{B} - \frac{C_4}{C}\right]. \qquad \text{...(14)}$$

From eqns. (13) and (14), we obtain

$$B = \frac{N^{1/(2K+1)}\,(S-1)\,(\alpha - t)^{-K}}{8\pi\eta LN(\alpha - t)^{(1-S)} + \beta(S-1)} \qquad \text{...(15)}$$

and

$$C = \frac{(S-1)\,(\alpha - t)^{1+K}}{N^{1/(2K+1)}\,[8\pi\eta LN(\alpha - t)^{(1-S)} + \beta(S-1)]} \qquad \text{...(16)}$$

where α, β and N are constants of integration and $S = 2K(K+1)$.

From eqns. (12), (15) and (16), we get

$$A = \frac{LN(S-1)\,(\alpha - t)^{-S}}{[8\pi\eta LN(\alpha - t)^{(1-S)} + \beta(S-1)]}. \qquad \text{...(17)}$$

Consequently, the line-element (1) takes the form

$$ds^2 = \left[\frac{(S-1)}{8\pi\eta LN(\alpha - t)^{(1-S)} + \beta(S-1)}\right]^2 \Big[L^2N^2(\alpha - t)^{-2S} \times (dx^2 - dt^2) + N^{2/(2K+1)}\,(\alpha - t)^{-2K}\,dy^2 + \frac{1}{N^{2/(2K+1)}}\,(\alpha - t)^{2(1+K)}\,dz^2\Big]. \qquad \text{...(18)}$$

By a suitable transformation of co-ordinates, the metric (18) is reduced to the form

$$ds^2 = \left[\frac{(S-1)\,T^{-S}}{8\pi\eta T^{(1-S)} + a(S-1)}\right]^2 [dX^2 - dT^2 + T^{2K(2K+1)}\,dY^2 + T^{2(K+1)(2K+1)}\,dZ^2] \qquad \text{...(19)}$$

where a is an arbitrary constant.

3. Some Physical Features

The distribution in the model is given by

$$
\begin{aligned}
8\pi p = {} & 16\pi\eta a\,[\tfrac{1}{3}(4K^2 + K - 2) + KT]\,T^{(S-1)} \\
& - \frac{320}{3}\pi^2\eta^2 - \tfrac{3}{2}S\left[\frac{8\pi\eta}{S-1} + aT^{(S-1)}\right]^2 \\
& + 8\pi\zeta\,[a(S-1)\,T^{(S-1)} - 16\pi\eta] - \Lambda \qquad \ldots(20)
\end{aligned}
$$

$$
\begin{aligned}
8\pi\epsilon = {} & 32\pi^2\eta^2\left[\frac{5S - 4(S^2+1)}{(S-1)^2}\right] + 96\pi^2\eta^2 - \tfrac{3}{2}\,a^2ST^{2(S-1)} \\
& + 8\pi\eta a\left[\frac{2S - 3(S^2+1)}{S-1}\right] T^{S-1} + \Lambda \qquad \ldots(21)
\end{aligned}
$$

The reality conditions (Ellis 1971)

(i) $\epsilon + p > 0$

and

(ii) $\epsilon + 3p > 0$

impose restrictions on the time during which the model exists. The flow vector V^i of the distribution is given by

$$V^1 = V^2 = V^3 = V_1 = V_2 = V_3 = 0$$

$$V^4 = \frac{8\pi\eta T}{(S-1)} + aT^S, \quad V_4 = \frac{(1-S)\,T^{-S}}{8\pi\eta T^{(1-S)} + a(S-1)}$$

Clearly, $V^i_{;j}\,V^j = 0$, so that the flow is geodesic.

The expressions for expansion θ, rotation ω and shear σ_{ij} calculated for the velocity vector V^i are given by

$$\left.\begin{aligned} \theta &= a(S-1)\,T^{(S-1)} - 16\pi\eta \\ \omega &= 0 \end{aligned}\right\} \qquad \ldots(22)$$

and

$$\left.\begin{aligned}
\sigma_{11} &= \frac{\tfrac{1}{3}(S-1)\,(5S - S^2 - 1)\,T^{-(S+1)}}{8\pi\eta T^{(1-S)} + a(S-1)} \\
\sigma_{22} &= \frac{\tfrac{1}{3}(S-1)\,(3K - S + 1)\,T^{(S-2K-1)}}{8\pi\eta T^{(1-S)} + a(S-1)} \\
\sigma_{33} &= -\frac{\tfrac{1}{3}(S-1)\,(3K + S + 2)\,T^{(S+2K+1)}}{8\pi\eta T^{(1-S)} + a(S-1)}
\end{aligned}\right\} \qquad \ldots(23)$$

the other components of the shear tensor σ_{ij} being zero. Hence,

$$\sigma^2 = \frac{1}{18}\left[\frac{8\pi\eta}{S-1} + aT^{(S-1)}\right]^2 [5S - S^2 - 1)^2 + (3K - S + 1)^2 + (3K + S + 2)^2]. \qquad ...(24)$$

The non-vanishing components of the conformal curvature tensor are

$$C^{12}_{12} = - C^{13}_{13} = - K(K+1)(2K+1)\left[\frac{8\pi\eta}{S-1} + aT^{(S-1)}\right]^2. \qquad ...(25)$$

Thus, the effect of viscosity is to prevent the shear and the free gravitational field from withering away for large values of T. It is also clear from eqn. (22) that the effect of viscosity is to retard the expansion of the model.

The metric (19) is conformal to the metric

$$ds^2 = dX^2 - dT^2 + T^{2K(2K+1)}\, dY^2 + T^{2(K+1)(2K+1)}\, dZ^2. \qquad ...(26)$$

The metric (26) represents a viscous fluid cosmological model in which the kinematic viscosity η_0 is $-\dfrac{K(K+1)}{4\pi T}$ and the distribution in the model is given by

$$8\pi p_0 = 8\pi\zeta\frac{(2K+1)^2}{T} - \frac{5K(K+1)(2K+1)^2}{3T^2} - \Lambda \qquad ...(27)$$

and

$$8\pi\epsilon_0 = \frac{K(K+1)(2K+1)^2}{T^2} + \Lambda. \qquad ...(28)$$

Reality conditions require that

$$-1 < K < 0 \qquad ...(29)$$

and ζ is greater than the greater among

$$\frac{K(K+1)}{12\pi T} \text{ and } \frac{K(K+1)}{6\pi T} + \frac{\Lambda T}{12\pi(2K+1)^2}. \qquad ...(30)$$

References

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