ON THE ABSOLUTE L-SUMMABILITY OF THE rTH DERIVED FOURIER SERIES

by B. PATRA, *Sinha Lodge, Emporium Lane, Cuttack* 7

(*Received* 16 *October* 1976; *after revision* 3 *May* 1977)

Absolute L-Summability was first investigated by Mohanty and Patnaik (1968). Subsequently it was studied by Ray (1972) for the first derived series of a Fourier series. In the present paper we study the ablolute L-summability of rth derived Fourier Series.

§1. *Definition* — Let $f(t) \in L(-\pi, \pi)$ and periodic with period 2π. Its Fourier series is given by

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(t). \qquad ...(1.1)$$

Formally, the rth derived series of (1.1) at $t = \theta$ is

$$\sum_{n=1}^{\infty} \left(\frac{d}{d\theta}\right)^r A_n(\theta). \qquad ...(1.2)$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(\theta + t) + f(\theta - t)\}$$

$$\psi(t) = \tfrac{1}{2}\{f(\theta + t) - f(\theta - t)\}$$

$$P(t) = \sum_{i=0}^{r-1} \frac{\mu_i t^i}{i\,!}, \text{ where } \mu\text{'s are arbitrary.}$$

$$X(t) = \tfrac{1}{2}[\{f(\theta + t) - P(t)\} + (-1)^r \{f(\theta - t) - P(-t)\}]$$

$$g(t) = \frac{r\,!\,X(t)}{t^r}$$

$$g_1(t) = \frac{1}{\log \frac{k}{t}} \int_t^{\pi} \frac{g(u)}{u}\, du, \text{ where } k > \pi.$$

§2. As in Borwein (1958), we say that an infinite sequence $\{S_n\}$ is summable L, if

$$L(x) = \frac{-1}{\log(1-x)} \sum_{n=1}^{\infty} S_n \frac{x^n}{n}$$

tends to a finite limit S as $x \to 1$ in an open interval $(0, 1)$.

If $L(x)$ is of bounded variation in $(\delta, 1)$ for some δ satisfying $0 < \delta < 1$, then we say (Mohanty and Patnaik 1968) that $\{S_n\}$ is absolutely summable L or $| L |$. If P and Q are methods of summability, Q is said to include P (written as $P \Rightarrow Q$) if every series summable by the method P is summable also by the method Q.

§3. Ray (1972) has proved that if $g_1(t) \in B.V.\ (0, \pi)$ and $g(t) \in L(0, \pi)$, then the first derived series is summable $| L |$. We now prove the following.

Theorem 1 — If $g(t) \in L(0, \pi)$ and $g_1(t) \in B.V.\ (0, \pi)$, then the series (1.2) is summable $| L |$ for every r.

We need the following lemmas to prove the theorem.

Since $| C | \Rightarrow | A |$ (Fekete 1932-33), and

$| A | \Rightarrow | L |$ (Mohanty and Patnaik 1968, Lemma 3)

we are now able to state,

Lemma 1 — $| C | \Rightarrow | L |$.

Lemma 2 — If $\delta > 0$

(i) $$\sum \frac{2}{\pi} \int_0^{\pi} \tfrac{1}{2}\{P(t) + (-1)^r P(-t)\}\,(d/dt)^r \cos nt\, dt$$

is summable $| C, r-1+\delta |$ or $| C, r+\delta |$ according as r is even or odd.

(ii) $\sum \int_0^{\pi} \lambda(t)\, t^r (d/dt)^r \cos nt\, dt$ is summable $| C, r+\delta |$ when $\delta > 0$ and $\lambda(t) \in B.V.\ (0, \pi)$.

This is a restatement of a result contained in the proof of a theorem given by Hyslop (1939) [see $\Sigma a_{n,1}$, $\Sigma a_{n,2}$ (for the case even and odd of (i) respectively) and Σa_n (for (ii)) of Hyslop's theorem 1].

Lemma 3 — $$(d/dt)^r \left\{ \cot \tfrac{1}{2}t \tan^{-1} \frac{x \sin t}{1 - x \cos t} + \tfrac{1}{2} \log (1 - 2x \cos t + x^2) \right\} =$$

(*equation continued on p.* 730)

$$= \begin{cases} O\left\{\frac{1}{t^r(1-x)} + \sum_{i=1}^{r} \frac{1}{t^{r+1-i}} \cdot \frac{1}{(1-x)^i}\right\} + O\left\{\frac{1}{(1-x)^r}\right\}, \text{ when } t < 1-x \\ O\left\{\frac{1}{t^{r+1}}\right\} \quad \text{when } t > 1-x. \end{cases}$$

PROOF OF LEMMA 3 : The left-hand side of Lemma 3 can be written as

$$(d/dt)^r \left\{ \cot \tfrac{1}{2}t \tan^{-1} \frac{x \sin t}{1 - x \cos t}\right\} + (d/dt)^r \{\tfrac{1}{2} \log (1 - 2x \cos t + x^2)\}$$
$$= P + Q, \text{ say.}$$

First we estimate P.

$$\text{Let } u = u_0 = \cot \tfrac{1}{2}t = O\left(\frac{1}{t}\right) \quad \text{...(3.1)}$$

and

$$v = v_0 = \tan^{-1} \frac{x \sin t}{1 - x \cos t} = \begin{cases} O\left(\frac{t}{1-x}\right) \\ O(1) \end{cases} \quad \text{...(3.2)}$$

$$u_0 \sin \tfrac{1}{2}t = \cos \tfrac{1}{2}t \quad \text{...(3.3)}$$

Differentiating (3.3), we get

$$u_1 \sin \tfrac{1}{2}t + \tfrac{1}{2}u_0 \sin\left(\tfrac{1}{2}t + \frac{\pi}{2}\right) = \tfrac{1}{2}\cos\left(\tfrac{1}{2}t + \frac{\pi}{2}\right). \quad \text{...(3.4)}$$

Using (3.1) and (3.4)

$$u_1 = O\left(\frac{1}{t^2}\right). \quad \text{...(3.5)}$$

Now we proceed to establish the following estimate by induction,

$$u_r = O\left(\frac{1}{t^{r+1}}\right). \quad \text{...(3.6)}$$

Let S_k stand for the statement "$u_0, u_1, \ldots,$ and u_k hold". From (3.1) and (3.3), it follows that S_0 and S_1 are true statements.

Let the statement S_k hold for any k. Now we proceed to show S_{k+1} holds, i.e., to prove $u_{k+1} = O\left(\frac{1}{t^{k+1}}\right)$.

Applying Leibniz's theorem to the relation (3.3) for $(k+1)$th derivative, we get

$$u_{k+1} \sin \tfrac{1}{2}t + \binom{k}{1} \tfrac{1}{2}\, u_k \sin \tfrac{1}{2}(t+\pi) + \binom{k}{2} \frac{1}{2^2}\, u_{k-1} \sin \tfrac{1}{2}(t+2\pi) + \dots$$
$$+ \binom{k}{j} \frac{1}{2^j}\, u_{k-j+1} \sin \tfrac{1}{2}(t+j\pi) + \dots + \frac{1}{2^k}\, u_0 \sin \tfrac{1}{2}(t+k\pi)$$
$$= \frac{1}{2^k} \cos \tfrac{1}{2}(t+k\pi)$$

$$\therefore\ u_{k+1} = \frac{-1}{\sin \frac{1}{2}t} \left\{ \binom{k}{1} \tfrac{1}{2}\, u_k \sin \tfrac{1}{2}(t+\pi) + \binom{k}{2} \frac{1}{2^2}\, u_{k-1} \sin (t+2\pi) \right.$$
$$+ \dots + \binom{k}{j} \frac{1}{2^j}\, u_{k-j+1} \sin \tfrac{1}{2}(t+j\pi) + \dots + \frac{1}{2^k}\, u_0 \sin \tfrac{1}{2}(t+k\pi)$$
$$\left. - \frac{1}{2^k} \cos \tfrac{1}{2}(t+k\pi) \right\}$$

$u_{k+1} = O\left(\frac{1}{t^{k+2}}\right)$ since S_k holds for arbitrary k.

$$\therefore\ u_r = O\left(\frac{1}{t^{r+1}}\right). \qquad \dots(3.7)$$

Differentiating (3.2)

$$v_1 = \frac{x(\cos t - x)}{\Delta(x, t)}, \text{ where } \Delta(x, t) = 1 - 2x \cos t + x^2 = (1-x)^2 + 4x \sin^2 \tfrac{1}{2}t. \qquad \dots(3.8)$$

$$\therefore\ v_1 = \begin{cases} O\left\{\dfrac{1}{(1-x)}\right\} & \text{if } t < 1-x \\ O\left(\dfrac{1}{t}\right) & \text{if } t > 1-x. \end{cases}$$

By similar arguments as in the estimation of u_r, we have

$$v_r = \begin{cases} O\left\{\dfrac{1}{(1-x)^r}\right\}, & \text{if } t < 1-x \\ O\left(\dfrac{1}{t^r}\right), & \text{if } t > 1-x, \text{ when } r > 0, \end{cases} \qquad \dots(3.9)$$

$$v_r = \begin{cases} O\left\{\dfrac{t}{(1-x)}\right\} & \\ O(1) & \text{when } r = 0. \end{cases}$$

Now applying Leibniz's theorem for rth derivative of

$$(d/dt)^r \left\{ \cot \tfrac{1}{2}t \tan^{-1} \frac{x \sin t}{1 - x \cos t} \right\} = (d/dt)^r (u_0 v_0)$$

(*equation continued on p. 732*)

$$= \sum_{i=1}^{r} \binom{r}{i} u_{r-i} v_i + u_r v_0$$

$$= \begin{cases} \sum_{i=1}^{r} \binom{r}{i} O\left(\frac{1}{t^{r+1-i}}\right) \left\{\frac{1}{(1-x)^i}\right\} + O\left\{\frac{1}{t^r(1-x)}\right\} \text{ when } t < 1-x \\ \sum_{i=1}^{r} \binom{r}{i} O\left(\frac{1}{t^{r+1-i}}\right)\left(\frac{1}{t^i}\right) + O\left(\frac{1}{t^{r+1}}\right) O(1) \text{ when } t > 1-x. \end{cases}$$

$$= \begin{cases} O\left\{\sum_{i=1}^{r} \binom{r}{i} \frac{1}{(1-x)^i \, t^{r+1-i}} + \frac{1}{t^r(1-x)}\right\} \text{ when } t < 1-x \\ O\left(\frac{1}{t^{r+1}}\right) \text{ when } t > 1-x. \end{cases} \quad \ldots(3.10)$$

By similar arguments used in the estimation of P,

$$Q = \begin{cases} O\left\{\frac{1}{(1-x)^r}\right\} & \text{when } t < 1-x \\ O\left(\frac{1}{t^r}\right) & \text{when } t > 1-x. \end{cases} \quad \ldots(3.11)$$

Putting together the results (3.10) and (3.11) the lemma follows.

Lemma 4 — $(d/dt)^r \left\{\frac{x-\cos t}{\triangle(x,t)} + \cot \tfrac{1}{2}t \frac{\sin t}{\triangle(x,t)}\right\}$

$$= \begin{cases} O\left\{\frac{1}{(1-x)^{r+2}}\right\} \text{ when } t < 1-x \\ O\left(\frac{1}{t^{r+2}}\right) \quad \text{ when } t > 1-x. \end{cases}$$

PROOF OF LEMMA 4: $(d/dt)^r \left\{\frac{x-\cos t}{\triangle(x,t)} + \frac{1+\cos t}{\triangle(x,t)}\right\} = (d/dt)^r \left\{\frac{1+x}{\triangle(x,t)}\right\}$.

Now proceeding exactly in a similar manner as in the case of Lemma 3, we establish Lemma 4.

§4. *Proof of the Theorem* : If r is even

$$(d/dt)^r A_n(t) = \frac{2}{\pi} \int_0^\pi \phi(t) \, (d/dt)^r \cos nt \, dt$$

$$= \frac{2}{\pi} \int_0^\pi \tfrac{1}{2} \{P(t) + P(-t)\} \, (d/dt)^r \cos nt \, dt + \frac{2}{\pi} \int_0^\pi X(t) \, (d/dt)^r \cos nt \, dt$$

$$= \alpha_n + \beta_n, \text{ say.}$$

$$\beta_n = \frac{2}{\pi}\int_0^\pi X(t)\,(d/dt)^r \cos nt\,dt = \frac{2}{\pi}\int_0^\pi \frac{g(t)}{r!}\,t^r(d/dt)^r \cos nt\,dt$$

$$= \frac{2}{\pi r!}\int_0^\pi \frac{g(t)}{t}\,t^{r+1}\,(d/dt)^r \cos nt\,dt$$

$$= \frac{2}{\pi r!}\int_0^\pi (d/dt)\left\{g_1(t)\log\frac{k}{t}\right\} t^{r+1}(d/dt)^r \cos nt\,dt$$

$$= \frac{2}{\pi r!}\int_0^\pi dg_1(t)\log\frac{k}{t}\,t^{r+1}(d/dt)^r \cos nt - \frac{2}{\pi r!}\int_0^\pi g_1(t)\,t^r(d/dt)^r \cos nt\,dt$$

$= u_n + v_n$, say.

$$U_n = \sum_{\nu=0}^{n} u_\nu$$

$$= \frac{2}{\pi r!}\int_0^\pi dg_1(t)\,t^{r+1}\log\frac{k}{t}\,(d/dt)^r\left(\sum_{\nu=0}^{n}\cos\nu t\right)$$

$$= \frac{2}{\pi r!}\int_0^\pi dg_1(t)\,t^{r+1}\log\frac{k}{t}\,(d/dt)^r\,\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}$$

$$\sum_{n=1}^{\infty} U_n\frac{x^n}{n} = \frac{2}{\pi r!}\int_0^\pi dg_1(t)\,t^{r+1}\log\frac{k}{t}\,(d/dt)^r M(n,t),$$

where $M(n,t) = \sum_{n=1}^{\infty}\frac{x^n}{n}\,\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}$.

Now

$$M(n,t) = \operatorname{Im}\frac{1}{\sin\frac{1}{2}t}\sum_{n=1}^{\infty}\frac{x^n}{n}\,e^{i(n+(1/2))t} = \operatorname{Im}\frac{1}{\sin\frac{1}{2}t}\,e^{i(1/2)t}\sum_{n=1}^{\infty}\frac{x^n e^{int}}{n}$$

$$= -\operatorname{Im}\frac{e^{i(1/2)t}}{\sin\frac{1}{2}t}\log(1 - xe^{it})$$

$$= -\tfrac{1}{2}\log(1 - 2x\cos t + x^2) + \cot\tfrac{1}{2}t\,\tan^{-1}\frac{x\sin t}{1 - x\cos t}.$$

$$\therefore \sum_{n=1}^{\infty} \frac{U_n x^n}{n} = \frac{2}{\pi r!} \int_0^{\pi} dg_1(t)\, t^{r+1} \log \frac{k}{t} (d/dt)^r \Big\{ \tfrac{1}{2} \log (1 - 2x \cos t + x^2)$$

$$+ \cot \tfrac{1}{2}t \ \tan^{-1} \frac{x \sin t}{1 - x \cos t} \Big\}.$$

We now proceed to study the summability $| L |$ of the sequence U_n.

Before doing so, we write,

$$l_x = - \log (1 - x).$$

Assuming $\delta = 1 - e^{-1}$, we have

$$I = \int_{\delta}^{1} \left| (d/dx) \Big\{ l_x^{-1} \sum_{n=1}^{\infty} U_n \frac{x^n}{n} \Big\} \right| dx$$

$$\leqq \int_0^{\pi} | dg_1(t) | \left| \log \frac{k}{t}\, t^{r+1} \int_{\delta}^{1} \Big\{ \frac{1}{l_x} (d/dt)^r \sum_{n=1}^{\infty} \frac{x^n \sin (n + \frac{1}{2})t}{n} \Big\} \right| . \qquad \ldots(4.1)$$

Since by hypothesis $\int_0^{\pi} | dg_1(t) |$ is finite, it is sufficient to prove

$$L = \int_{\delta}^{1} \left| (d/dx) \Big[\frac{1}{l_x} (d/dt)^r \Big\{ - \tfrac{1}{2} \log (1 - 2x \cos t + x^2)\right.$$

$$\left. + \cot \tfrac{1}{2}t \tan^{-1} \frac{x \sin t}{1 - x \cos t} \Big\} \Big] \right| dx = O \Big\{ t^{-r-1} \Big(\log \frac{k}{t} \Big)^{-1} \Big\}. \qquad \ldots(4.2)$$

Now,

$$L \leqq \int_{\delta}^{1} \left| \frac{1}{(1 - x)\, l_x^2} (d/dt)^r \Big\{ - \tfrac{1}{2} \log (1 - 2x \cos t + x^2)\right.$$

$$\left. + \cot \tfrac{1}{2}t \ \tan^{-1} \frac{x \sin t}{1 - x \cos t} \Big\} \right| dx + \int_{\delta}^{1} \left| \frac{1}{l_x} (d/dt)^r \Big\{ - \frac{x - \cos t}{\triangle (x, t)}\right.$$

$$\left. + \cot \tfrac{1}{2}t \frac{\sin t}{\triangle (x, t)} \Big\} \right| dx$$

$$\leqq \int_{\delta}^{1} \frac{1}{(1-x)\, l_x^2} (d/dt)^r \left\{ \tfrac{1}{2} \log (1 - 2x \cos t + x^2) \right.$$

$$\left. + \cot \tfrac{1}{2}t \tan^{-1} \frac{x \sin t}{1 - x\cos t} \right\} dx + \int_{\delta}^{1} \frac{1}{l_x} (d/dt)^r \left\{ \frac{x - \cos t}{\triangle\,(x, t)} \right.$$

$$\left. + \cot \tfrac{1}{2}t \, \frac{\sin t}{\triangle\,(x, t)} \right\} dx.$$

$$= J + K \text{, say.}$$

Now, taking $\tau = 1 - \dfrac{t}{k}$

$$J = \int_{\delta}^{\tau} + \int_{\tau}^{1} = J_1 + J_2, \text{ say.}$$

By an appeal to Lemma 3,

$$J_1 = \int_{\delta}^{\tau} \frac{dx}{(1-x)\, l_x^2}\, O\left\{ \sum_{i=1}^{r} \left(\frac{1}{t^{r+1-i}}\right) \frac{1}{(1-x)^i} + \frac{1}{t^r(1-x)} \right\}$$

$$+ \int_{\delta}^{\tau} \frac{dx}{(1-x)\, l_x^2}\, O\left\{ \frac{1}{(1-x)^r} \right\}$$

$$= O\left\{ \sum_{i=1}^{r} \frac{1}{t^{r+1-i}} \int_{\delta}^{\tau} \frac{dx}{(1-x)^{i+1}\, l_x^2} + t^{-r} \int_{\delta}^{\tau} \frac{dx}{(1-x)^2\, l_x^2} \right\}$$

$$+ O\left\{ \int_{\delta}^{\tau} \frac{dx}{(1-x)^{r+1}\, l_x^2} \right\}$$

$$= O\left[\sum_{i=1}^{r} \frac{1}{t^{r+1-i}} \left\{ \frac{1}{(1-\tau)^i\, l_\tau^2} \right\} + O\left\{ \frac{1}{t^r(1-\tau)\, l_\tau^2} \right\} \right.$$

$$\left. + O\left\{ \frac{1}{(1-\tau)^r\, l_\tau^2} \right\} \right] = O\left\{ \sum_{i=1}^{r} t^{-r-1}\left(\log \frac{k}{t}\right)^{-2} + t^{-r-1}\left(\log \frac{k}{t}\right)^{-2} \right\}$$

$$+ O\left\{ t^{-r}\left(\log \frac{k}{t}\right)^{-2} \right\} = O\left\{ t^{-r-1}\left(\log \frac{k}{t}\right)^{-2} \right\} \qquad \ldots(4.3)$$

and

$$J_2 = \int_{\tau}^{1} \frac{dx}{(1-x)\, l_x^2}\, O(t^{-r-1}) = O(t^{-r-1}) \int_{\tau}^{1} \frac{dx}{(1-x)\, l_x^2}$$

$$= O\left(t^{-r-1} \frac{1}{l_\tau}\right) = O\left\{t^{-r-1} \left(\log \frac{k}{t}\right)^{-1}\right\}.$$

$$K = \int_{\delta}^{\tau} + \int_{\tau}^{1} = K_1 + K_2, \text{ say.} \qquad \ldots(4.4)$$

By virtue of Lemma 4

$$K_1 = \int_{\delta}^{\tau} \frac{dx}{l_x} O\left\{\frac{1}{(1-x)^{r+2}}\right\} = O\left(\int_{\delta}^{\tau} \frac{dx}{l_x(1-x)^{r+2}}\right)$$

$$= O\left[\left\{\frac{1}{l_x(1-x)^{r+1}}\right\}\right]_{\delta}^{\tau} \leqq O\left\{\frac{1}{l_\tau(1-\tau)^{r+1}}\right\} = O\left\{t^{-r-1}\left(\log \frac{k}{t}\right)^{-1}\right\} \qquad \ldots(4.5)$$

$$K_2 = \int_{\tau}^{1} \frac{dx}{l_x} O(t^{-r-2}) = O(t^{-r-2}) \int_{\tau}^{1} \frac{dx}{l_x}$$

$$= O(t^{-r-2}).\ O\left[\frac{1-x}{l_x}\right]_{\tau}^{1} = O\left\{t^{-r-1}\left(\log \frac{k}{t}\right)^{-1}\right\}. \qquad \ldots(4.6)$$

Taking the results (4.3), (4.4), (4.5) and (4.6) together, (4.2) is established.

By Lemma 2(i) it can be proved that $\Sigma\, \alpha_n$ is $|\, C, r-1+\delta\, |$, $\delta > 0$ and by virtue of Lemma 1 it is summable $|\, L\, |$.

By Lemma 2(ii) it can be proved that $\Sigma\, v_n$ is summable $|\, C, r+\delta\, |$, $\delta > 0$ and by Lemma 1 it is summable $|\, L\, |$.

Similar treatments are adopted for the case when r is odd.

This completes the proof of the theorem.

Acknowledgement

The author is indebted to Dr. B. K. Ray for suggesting this problem. He is also thankful to the referee for his suggestions for the improvement of this paper.

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