ON THE "LITTLEWOOD CONJECTURE"

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If we define $F(D) = q_0$, for all α, β real, there exists a $q \leqslant q_0$ such that $q \| q\alpha \| \| q\beta \| < 1/D$ ($\| x \|$ dist. from x to nearest integer).

We prove the following :

Theorem — If $c > \sqrt{2}$, there exists $D_0(c)$ so that if $D > D_0(c)$, then $F(D) > \exp(\sqrt{D}/c)$.

Corollary — $D \geqslant 90$, $F(D) > \exp(\sqrt{D}/3)$.

Lemma 1 — Let $A(q) = \text{area}\{(x, y) \mid 0 \leqslant x, y \leqslant 1,\ q \| qx \| \| qy \| < 1/D\}$. Then

$$A(q) = 8/Dq\left(\tfrac{1}{2} + \ln\left(\frac{\sqrt{Dq}}{2}\right)\right).$$

PROOF : Choose arbitrary point $(p/q,\ r/q)$ in $[0, 1] \times [0, 1]$.

C has coordinates : $(p/q + 1/2q,\ r/q + 1/2q)$

B has coordinates : $(p/q + 1/\sqrt{Dq^3},\ r/q + 1/\sqrt{Dq^3})$

$$\text{Area I} = \tfrac{1}{2} \cdot \frac{1}{\sqrt{Dq^3}} \cdot \frac{1}{\sqrt{Dq^3}} = \frac{1}{2Dq^3}.$$

$$\text{Area II} = \int_{1/\sqrt{Dq^3}}^{1/2q} \frac{1}{Dq^3 x}\,dx = \frac{1}{Dq^3} \ln(x) \Big|_{1/\sqrt{Dq^3}}^{1/2q}$$

$$= \frac{1}{Dq^3} \ln\left(\frac{\sqrt{Dq}}{2}\right).$$

$$\text{Area (I + II + III)} = \tfrac{1}{2} \cdot \frac{1}{2q} \cdot \frac{1}{2q} = \frac{1}{8q^2}.$$

$$\text{Therefore Area}\left(\frac{\text{I + II}}{\text{I + II + III}}\right) = \frac{4}{Dq} + \frac{8}{Dq} \ln\left(\frac{\sqrt{Dq}}{2}\right).$$

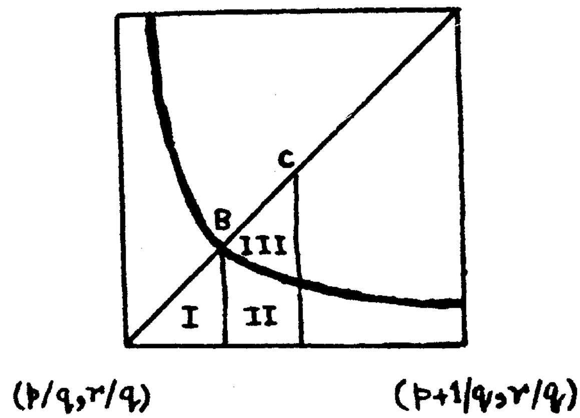

FIG. 1.

Now $\bigcup_{q=1}^{N} A(q) < \sum_{q=1}^{N} A(q)$ and so if we, for a fixed D, make $\sum_{q=1}^{N} A(q) < 1$ then surely $F(D) > N$.

Lemma 2 — $$\sum_{q=1}^{N} A(q) < \frac{8}{D}\left(\tfrac{1}{2} + \tfrac{1}{2}\ln(D) - \ln(2)\right)(1 + \ln(N))$$
$$+ \frac{4}{D}\left(\frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \frac{\ln^2(N)}{2} - \frac{\ln^2(3)}{2}\right)$$

PROOF :

$$\sum_{q=1}^{N} \frac{\ln(q)}{q} \leqslant \frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \sum_{q=4}^{N} \frac{\ln(q)}{q}$$
$$< \frac{\ln(q)}{2} + \frac{\ln(3)}{3} + \int_{3}^{N} \frac{\ln(x)}{x}\,dx$$
$$\leqslant \frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \frac{\ln^2(N)}{2} - \frac{\ln^2(3)}{2} \qquad \ldots(1)$$

$$\sum_{q=1}^{N} \frac{1}{q} \leqslant 1 + \int_{1}^{N} \frac{1}{x}\,dx = 1 + \ln(N). \qquad \ldots(2)$$

So

$$\sum_{q=1}^{N} A(q) = \frac{8}{D}\left(\tfrac{1}{2} + \tfrac{1}{2}\ln(D) - \ln(2)\right)\sum_{q=1}^{N}\frac{1}{q} + \frac{4}{D}\sum_{q=1}^{N}\frac{\ln(q)}{q}$$

$$< \frac{8}{D}\left(\tfrac{1}{2} + \tfrac{1}{2}\ln(D) - \ln(2)\right)(1 + \ln(N))$$

$$+ \frac{4}{D}\left(\frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \frac{\ln^2(N)}{2} - \frac{\ln^2(3)}{2}\right).$$

Lemma 3 — If $c > \sqrt{2}$, there exists $D_0(c)$ so that $\sum_{q=1}^{N} A(q) < 1$, with

$$N = \exp(\sqrt{D}/c).$$

PROOF: Set $N = \exp(\sqrt{D}/c)$, then $\ln(N) = \frac{\sqrt{D}}{c}$, $\ln^2(N) = D/c^2$, and so from Lemma 2

$$\sum_{q=1}^{N} A(q) < \frac{8}{D}\left(\tfrac{1}{2} + \tfrac{1}{2}\ln(D) - \ln(2)\right)\left(1 + \frac{\sqrt{D}}{c}\right)$$

$$+ \frac{4}{D}\left(\frac{\ln(2)}{2} + \frac{\ln(3)}{3} - \frac{\ln^2(3)}{2}\right) + \frac{2}{c^2} < 1.$$

Multiply both sides by D and we have

$$8\left(\tfrac{1}{2} + \tfrac{1}{2}\ln(D) - \ln(2)\right)\left(1 + \frac{\sqrt{D}}{c}\right)$$

$$+ 4\left(\frac{\ln(2)}{2} + \frac{\ln(3)}{3} - \frac{\ln^2(3)}{2}\right) < \frac{c^2-2}{c^2}D.$$

Choose D_0 so that $\frac{8}{c}\ln(D_0)\sqrt{D_0} < \frac{c^2-2}{c^2}D_0$ or $\frac{8}{c}\ln(D_0) < \frac{c^2-2}{c^2}\sqrt{D_0}$

or $\frac{8c}{c^2-2}\ln(D_0) < \sqrt{D_0}$.

As an example $D_0 = \left[\max\left\{\frac{8c}{c^2-2}, 9\right\}\right]^4$ will work.

The corollary follows on setting $c = 3$ and computing $D_0(3)$. It can be taken $D_0(3) = 90$.

As previously announced in January 1975 Notices of Am. Math. Soc. $F(1) = F(2) = F(3) = F(4) = 1$, $F(5) = F(6) = 2 = F(7)$, $F(8) = 3$, $F(9) = 7$, $F(10) = 15$ and $F(11) = 49$. The estimate of the theorem is surely very poor since no overlapping of area was considered