ON A SPECIFIED AFFINE CONNEXION IN AN ALMOST CONTACT MANIFOLD

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(*Received* 16 *May* 1975)

In the present paper the authors have defined an affine connexion and studied its properties in an almost contact manifold. We have also obtained curvature tensor of this affine connexion.

1. INTRODUCTION

Let us consider an n-dimensional real differentiable manifold V_n $(n = 2m + 1)$ of class C^∞. Let there exist in V_n a C^∞ vector valued linear function F, a C^∞ vector field T and a C^∞ 1-form A satisfying

$$F(F(X)) = -X + A(X)\,T \qquad \ldots(1.1)$$

which implies

$$\text{rank}\,(F) = n - 1,\ F(T) = 0, \quad A(F(X)) = 0 \qquad \ldots(1.1a)$$

$$A(T) = 1 \text{ and } n \text{ is odd} = 2m + 1 \text{ (say)}$$

for arbitrary vector field X. Then V_n is called an almost contact manifold and the structure (F, T, A) is called an almost contact structure.

Agreement 1.1—In what follows, the equations containing X, Y, Z, U, V will hold for arbitrary vector fields X, Y, Z, U, V.

Let D be an affine connexion in V_n and let S, a vector valued bilinear function, be its torsion tensor:

$$S(X, Y) = D_X Y - D_Y X - [X, Y]$$

where [ ] denotes Lie bracket.

2. AFFINE CONNEXION

We will assume that the affine connexion D satisfies the following properties:

$$(D_X A)(Y) + (D_Y A)(X) = 0 \qquad \ldots(2a)$$

$$(D_X F)(Y) + (D_{\bar{X}} F)(\bar{Y}) = 0 \qquad \ldots(2b)$$

$$D_X T = 0. \qquad \ldots(2c)$$

Theorem 2.1—Equation (2*b*) implies

$$D_{\bar{X}}\bar{Y} + \overline{D_X\bar{Y}} - \overline{D_{\bar{X}}Y} + D_XY = X\big(A(Y)\big)\,T \qquad \ldots(2.1a)$$

$$D_{\bar{X}}\bar{Y} + \overline{D_X\bar{Y}} - \overline{D_{\bar{X}}Y} + D_XY = A(D_XY + D_{\bar{X}}\bar{Y})\,T \qquad \ldots(2.1b)$$

$$\overline{D_{\bar{X}}\bar{Y}} - D_X\bar{Y} + D_{\bar{X}}Y + \overline{D_XY} = \{FX\big(A(Y)\big)\}\,T \qquad \ldots(2.1c)$$

$$D_{\bar{X}}\bar{Y} + \overline{D_X\bar{Y}} - \overline{D_{\bar{X}}Y} + D_XY = 0. \qquad \ldots(2.1d)$$

PROOF: Operating F on Y in (2*b*) and using (1·1) and (2*c*) we obtain (2.1*a*). Operating F on (2.1*a*) and using (1.1), we get (2.1*b*). Operating F on X in (2.1*a*) and using (1.1) we obtain (2.1*c*). Barring (2*b*) and using (1.1) we get (2.1*d*).

Theorem 2.2—In an almost contact manifold V_n, we have

$$(D_XA)(Y) = A(D_{\bar{X}}\bar{Y}). \qquad \ldots(2.2a)$$

PROOF: Comparing (2.1*a*) and (2.1*b*) we get (2.2*a*).

Corollary 2.1—We have

$$(D_XA)(T) = 0 \qquad \ldots(2.2b)$$

$$(D_TA)(Y) = 0. \qquad \ldots(2.2c)$$

PROOF: Putting T for Y in (2.2*a*) and using (1.1) we get (2.2*b*). Putting T for X in (2.2*a*) and using (1.1) we get (2.2*c*).

Corollary 2.2—We have

$$A(D_T\bar{Y}) = 0. \qquad \ldots(2\cdot2d)$$

PROOF: From (1.1) we have

$$(D_TA)(\bar{X}) + A(D_T\bar{X}) = 0.$$

Using (2.2*c*) in this equation, we have (2.2*d*).

Theorem 2.3—We have

$$D_T\bar{Y} = \overline{D_TY}. \qquad \ldots(2.3a)$$

PROOF : Putting T for X in (2*b*) and using (1.1) we get (2.3*a*)

Corollary 2.3—In V_n, we get

$$A(D_{\bar{X}}\bar{Y} + D_{\bar{Y}}\bar{X}) = 0. \qquad \ldots(2.3b)$$

PROOF : Operating F on X and Y in (2*a*) and using (1.1*a*) we get (2.3*b*).

Theorem 2.4—In an almost contact manifold V_n, we have

$$\overline{S(T, X)} + \overline{[T, X]} + S(\bar{X}, T) + [\bar{X}, T] = 0 \qquad \ldots(2.4)$$

and

$$S(T, X) + [T, X] + \overline{S(T, \bar{X})} + [T, \bar{X}] = A(D_T X)T. \qquad \ldots(2.5)$$

PROOF: Torsion tensor S of the connexion D is given by

$$S(X, Y) = D_X Y - D_Y X - [X, Y].$$

Using (2c) in this, we get

$$S(T, X) = D_T X + [X, T] \qquad \ldots(2.6)$$

and

$$S(T, \bar{X}) = D_T \bar{X} - [T, \bar{X}]. \qquad \ldots(2.7)$$

Using (2.6), (2.7) and (2.3a) and the fact that the torsion tensor S is skew-symmetric, we get (2.4). Operating F on (2.4) and using (1.1) and (2c), we get (2.5)

Theorem 2.5—Let

$$'F(X, Y) \overset{def}{=\!=} (D_X A)(Y) \qquad \ldots(2.8)$$

then $'F$ is skew-symmetric and

$$'F(\bar{X}, \bar{Y}) + 'F(X, Y) = 0. \qquad \ldots(2.9)$$

PROOF: From (2a) it follows that $'F$ is skew-symmetric in X and Y, *i.e.*, $'F(X, Y) = -\,'F(Y, X)$.

Now (2.8) gives

$$'F(\bar{X}, \bar{Y}) = (D_{\bar{X}} A)(\bar{Y})$$

or

$$'F(\bar{X}, \bar{Y}) = \bar{X}\big(A(\bar{Y})\big) - A(D_{\bar{X}} \bar{Y}).$$

Using (1.1a) we get

$$'F(\bar{X}, \bar{Y}) = -A(D_{\bar{X}} \bar{Y}).$$

Now, using (2.1a), we have

$$\begin{aligned} 'F(\bar{X}, \bar{Y}) &= -A\big(-\overline{D_X \bar{Y}} + \overline{D_{\bar{X}} Y} - D_X Y + X(A(Y))\,T\big) \\ &= A(D_X Y) - X\big(A(Y)\big) \\ &= -(D_X A)\,Y = -\,'F(X, Y). \end{aligned}$$

i.e.,

$$'F(\bar{X}, \bar{Y}) + 'F(X, Y) = 0.$$

Corollary 2.4—We have

$$'F(T, X) = 0 \qquad \text{...(2.8a)}$$

and

$$'F(X, \bar{Y}) + 'F(\bar{X}, Y) = 0. \qquad \text{...(2.8b)}$$

PROOF: From (2.8) and (2.2*a*) we get (2.8*a*). Operating F on X in (2.9) and using (1.1) we get (2.9*a*).

Theorem 2.6—Let N, a vector valued bilinear function, be the Nijenhuis tensor (Mishra 1972)

$$N(X, Y) \overset{def}{=\!=} D_{\bar{X}}\bar{Y} - D_{\bar{Y}}\bar{X} - D_X Y + D_Y X + A(D_X Y - D_Y X)\, T$$

$$- \overline{D_X \bar{Y}} + \overline{D_{\bar{Y}} X} - \overline{D_{\bar{X}} Y} + \overline{D_Y \bar{X}}. \qquad \text{...(2.10)}$$

Then, we have

$$A\big(N(X, Y)\big) = 2\,'F(X, Y). \qquad \text{...(2.11)}$$

PROOF: Using (2.1*a*) in (2.10) we have

$$N(X, Y) = 2D_Y X - 2D_X Y + 2\ (\overline{D_Y \bar{X}} - \overline{D_X \bar{Y}})$$

$$+ 2A(D_X Y - D_Y X)\, T + \{(D_X A)(Y) - (D_Y A)(X)\}\, T.$$

Using (2.8) and the fact that

$$2F\big((D_Y F)\, X - (D_X F)\, Y\big) = 2D_Y X - D_X Y + 2\,(\overline{D_Y \bar{X}} - \overline{D_X \bar{Y}})$$

$$+ 2A(D_X Y - D_Y X)\, T$$

we have,

$$N(X, Y) = 4F\big((D_Y F)\, X\big) + 2\,\big('F(X, Y)\, T\big).$$

Now, operating A on the above, we get

$$A\big(N(X, Y)\big) = 2\,'F(X, Y).$$

Corollary 2.5—We have

$$A\big(N(\bar{X}, \bar{Y})\big) = -A\big(N(X, Y)\big). \qquad \text{...(2.12)}$$

PROOF: Operating F on X and Y in (2.11) and using (2.9) we have (2.12).

Corresponding to the Nijenhuis tensor of an almost complex manifold (Yano 1965), there are three other tensors in V_n, P which is a scalar-valued bilinear function, Q which is a vector valued linear function and R which is a 1-form P, Q, R are given by

$$P(X, Y) \stackrel{def}{=} (D_Y A)(\bar{X}) - (D_{\bar{X}} A)(Y) + (D_{\bar{Y}} A)(X) - (D_X A)(\bar{Y})$$

$$Q(X) \stackrel{def}{=} [T, \bar{X}] + S(T, \bar{X}) + \overline{[X, T]} + S\overline{(X, T)}$$

and

$$R(X) \stackrel{def}{=} (D_X A)(T) - (D_T A)(X).$$

Theorem 2.7—For an almost contact manifold V_n, we have

$$P(X, Y) = 4\,'F(Y, \bar{X}) \qquad \ldots(2.13)$$

$$Q(X) = 0 \qquad \ldots(2.14)$$

$$R(X) = 0. \qquad \ldots(2.15)$$

PROOF: Using the definitions of P and $'F$ and Corollary (2.4), we have (2.13). Using (2.4) in the definition of $Q(X)$ we get (2.14). Also, by virtue of (2.2*b*) and (2.2*c*), we have (2.15).

Thus, we see that in V_n, Q and R are identically vanishing. Let us suppose that $N = 0$, identically. Then (2.11) gives

$$2\,'F(X, Y) = 0.$$

Consequently $P = 0$. Hence we have the following.

Theorem 2.8—In an almost contact manifold V_n, the vanishing of Nijenhuis tensor N means vanishing of P, Q, R.

3. Curvature Tensor

Let K be the curvature tensor with respect to the connexion D:

$$K(X, Y, Z) \stackrel{def}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \qquad \ldots(3.1)$$

and Ric be the corresponding Ricci tensor given by

$$\text{Ric}(Y, Z) \stackrel{def}{=} (C_1^1 K)(Y, Z)$$

where C_1^1 is contraction in first slot.

If B is a 1-form then

$$(D_X D_Y B - D_Y D_X B - D_{[X, Y]} B)(Z) = -B(K(X, Y, Z)). \qquad \ldots(3.2)$$

Theorem 3.1—In an almost contact manifold V_n as defined above, we always have

$$K(X, Y, T) = 0 \quad \text{and hence} \quad \text{Ric}(Y, T) = 0. \qquad \ldots(3.3)$$

PROOF: Putting T for Z in (3.1) and using (2*c*), *i.e.*, $D_X T = 0$, the above equation (3.1) reduces to

$$K(X, Y, T) = 0.$$

Theorem 3.2—In an almost contact manifold, we have

$$'F\big(S(X, Y), Z\big) + (D_X{}'F)(Y, Z) - (D_Y{}'F)(X, Z) + A\big(K(X, Y, Z)\big) = 0. \quad ...(3.4)$$

PROOF: Taking covariant derivative of (2.8), we have

$$(D_X D_Y A)(Z) = 'F(D_X Y, Z) + (D_X{}'F)(Y, Z). \quad ...(3.4a)$$

Interchanging X and Y in (3.4a) we have

$$(D_Y D_X A)(Z) = 'F(D_Y X, Z) + (D_Y{}'F)(X, Z), \quad ...(3.4b)$$

and putting $[X, Y]$ for X and Z for Y in (2.8), we get

$$(D_{[X,Y]}A)(Z) = 'F([X, Y], Z). \quad ...(3.4c)$$

Subtracting the sum of (3.4b) and (3.4c) from (3.4a) and using (3.2), we get (3.4).

Corollary—In an almost contact manifold V_n, $'F$ satisfies

$$\begin{aligned}&'F\big(S(X, Y), Z\big) + 'F\big(S(Y, Z), X\big) + 'F\big(S(Z, X), Y\big)\\ &\quad + A\big((D_X S)(Y, Z) + (D_Y S)(Z, X) + (D_Z S)(X, Y)\big)\\ &\quad + A\big(S(S(X, Y), Z) + S(S(Y, Z), X) + S(S(Z, X), Y)\big)\\ &\quad + 2\big((D_X{}'F)(Y, Z) + (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y)\big) = 0. \quad ...(3.5)\end{aligned}$$

PROOF: Taking cyclic permutation of X, Y, Z in (3.4) and adding all the three equations, we get,

$$\begin{aligned}&'F\big(S(X, Y), Z\big) + 'F\big(S(Y, Z), X\big) + 'F\big(S(Z, X), Y\big)\\ &\quad + A\big(K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y)\big)\\ &\quad + 2\big((D_X{}'F)(Y, Z) + (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y)\big) = 0. \quad ...(3.5a)\end{aligned}$$

Using Bianchi's first identity in (3.5a), we get (3.5).

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