SOME GENERATING FUNCTIONS IN UNIFIED FORM FOR THE CLASSICAL ORTHOGONAL POLYNOMIALS AND BESSEL POLYNOMIALS

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(*Received* 26 *June* 1975)

Using the approach of Fujiwara-Thakare we establish here generating functions for the classical orthogonal polynomials and Bessel polynomials. In the sequel we also obtain Carlitz type generating functions for the polynomials of Jacobi and Bessel.

1. INTRODUCTION

The purpose of this paper is to obtain some generating functions for the classical orthogonal polynomials—namely Jacobi polynomials, Laguerre polynomials, Hermite polynomials in the unified form and to show that these generating functions in turn yield, following the approach of Fujiwara (1966) and Thakare (1972, 1974, 1975*a*, *b*) the generating functions for the classical orthogonal polynomials and the Bessel polynomials.

It may be observed that Fujiwara (1966), Thakare (1972, 1974, 1975*a*, *b*), Karande and Thakare (1974), Patil and Thakare (1975) have succeeded in a great detail to unify the study of the classical orthogonal polynomials.

Thakare (1972) has stated the extended Jacobi polynomials $F(\alpha, \beta; x)$ by the following Rodrigues' formula:

$$F_n(\alpha, \beta; x) = \frac{1}{K_n W(x)} D^n [W(x) \{X(x)\}^n], \qquad D \equiv \frac{d}{dx} \qquad \ldots(1.1)$$

where the weight function $W(x)$, the quadratic function $X(x)$ and the constants K_n are respectively given by

$$W(x) = \frac{(x-a)^\alpha (b-x)^\beta}{(b-a)^{\alpha+\beta+1} B(\alpha+1, \beta+1)}; \quad (\alpha > -1, \quad \beta > -1). \ldots(1.2)$$

$$X(x) = c(x-a)(b-x) \text{ with } c > 0 \qquad \ldots(1.3)$$

and

$$K_n = (-1)^n . n!. \qquad \ldots(1.4)$$

In our discussion a parameter λ which is connected by the relation

$$\lambda = c(b - a) \text{ frequently occurs,} \qquad \ldots(1.5)$$

Thakare (1972) has stated the following relationships with these polynomials:

Case (A): Jacobi Polynomials

When

$$-a = b = \lambda = 1.$$

$$F_n(\alpha, \beta; x) = P_n^{(\beta, \alpha)}(x). \qquad \ldots(1.6)$$

Case (B): Laguerre Polynomials

When $a = 0$, $\beta = b$ and $\lambda = 1$, we have,

$$\lim_{b \to \infty} F_n(\alpha, b; x) = (-1)^n L_n^{(\alpha)}(x). \qquad \ldots(1.7)$$

Case (C): Hermite Polynomials

For $\beta = \alpha$, $-a = b = \sqrt{\alpha}$, $(\alpha > 0)$ and in view of $\lambda \to \frac{2}{\sqrt{\alpha}}$, we get,

$$\lim_{\sqrt{\alpha} \to \infty} F_n(\alpha, \alpha; x) = \frac{H_n(x)}{n!}. \qquad \ldots(1.8)$$

Case (D): Bessel polynomials

Following Agarwal (1954), Thakare (1972) stated the relationship between Bessel polynomials $y_n(x; r, s)$ and the extended Jacobi polynomials $F_n(\alpha, \beta; x)$.

Put $-a = b = \lambda = 1$, $\alpha = r - \epsilon - 1$, $\beta = \epsilon - 1$ and replace x by $\left(1 + \frac{2x\epsilon}{s}\right)$

to get,

$$\lim_{\epsilon \to \infty} \frac{\Gamma(n+1)}{\epsilon^n} F_n\left(r - \epsilon - 1,\ \epsilon - 1; 1 + \frac{2x\epsilon}{s}\right) = y_n(x; r, s). \qquad \ldots(1.9)$$

In the sequel we also obtain Carlitz-type (1961) generating function for the extended Jacobi polynomials, and as limiting cases those for the classical orthogonal polynomials and the Bessel polynomials.

2. Generating Function

In this section, we obtain a generating function for the extended Jacobi polynomials, the method of obtaining the generating function is essentially that of Brafman (1951).

Thakare (1972) has stated,

$$F_n(\alpha, \beta; x) = \lambda^n \left(\frac{x-a}{b-a}\right)^n \binom{n+\beta}{n} {}_2F_1 \left[\begin{matrix} -n, & -n-\alpha; & \\ & & \frac{x-b}{x-a} \\ & 1+\beta; & \end{matrix}\right]. \qquad \ldots(2.1)$$

Using (2.1) we consider,

$$\sum_{n=0}^{\infty} \frac{(\delta)_n F_n(\alpha, \beta; x) t^n}{(1+\alpha)_n (1+\beta)_n}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\delta)_{n+k}}{n!\, k!\, (1+\alpha)_n (1+\beta)_k} \left(\frac{\lambda t(x-a)}{b-a}\right)^n \left(\frac{\lambda t(x-b)}{b-a}\right)^k$$

$$= \psi_2\left(\delta; 1+\beta, 1+\alpha; \frac{\lambda t(x-a)}{b-a}, \frac{\lambda t(x-b)}{b-a}\right), \qquad \ldots(2.2)$$

where ψ_2 is Horn's function of two arguments (Erdelyi 1953, p. 225).

Particular Cases

(I) For $-a = b = \lambda = 1$ we get the result obtained by Bhonsle (1962)

$$\psi_2\left(a; 1+\alpha, 1+\beta; \frac{(x-1)t}{2}, \frac{(x+1)t}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n P_n^{(\alpha,\beta)}(x) t^n}{(1+\alpha)_n (1+\beta)_n}. \qquad \ldots(2.3)$$

(II) When $\delta = 1+\beta$, $\beta = b$, $\lambda = 1$, $a = 0$ and on account of (1.7) we get the usual generating function for the Laguerre polynomials,

$$\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x) t^n}{(1+\alpha)_n} = e^t {}_0F_1 \left[\begin{matrix} -\,; & \\ & -xt \\ 1+\alpha; & \end{matrix}\right]. \qquad \ldots(2.4)$$

3. More Generating Functions

Thakare (1972) has also stated the extended Jacobi polynomials as:

$$F_n(\alpha, \beta; x) = \lambda^n \binom{n+\beta}{n} {}_2F_1 \left[\begin{matrix} -n, & 1+\alpha+\beta+n; & \\ & & \frac{b-x}{b-a} \\ & 1+\beta; & \end{matrix}\right]. \qquad \ldots(3.1)$$

Patil and Thakare (1975) have also proved the generating function for these polynomials, which runs as:

$$2^{\alpha+\beta} R^{-1} (1 + \lambda t + R)^{-\alpha} (1 - \lambda t + R)^{-\beta} = \sum_{n=0}^{\infty} F_n(\alpha, \beta; x)\, t^n \qquad ...(3.2)$$

where

$$R = \left[1 + \frac{2\lambda t\,(a + b - 2x)}{b - a} + \lambda^2 t^2\right]^{\frac{1}{2}}. \qquad ...(3.3)$$

Now consider

$$\frac{d}{dt}[t^\gamma (1 + \lambda t + R)^{-\gamma} (1 - \lambda t + R)^{-\gamma}]$$

$$= \gamma\, t^{\gamma-1} (1 + \lambda t + R)^{-\gamma} (1 - \lambda t + R)^{-\gamma}$$

$$\times \left[1 - \frac{t\,(R' - \lambda)}{1 - \lambda t + R} - \frac{t\,(R' + \lambda)}{1 + \lambda t + R}\right], \qquad R' = \frac{dR}{dt}.$$

On simplifying we get,

$$\frac{d}{dt}[t^\gamma (1 + \lambda t + R)^{-\gamma} (1 - \lambda t + R)^{-\gamma}]$$

$$= \gamma\, R^{-1} t^{\gamma-1} (1 + \lambda t + R)^{-\gamma} (1 - \lambda t + R)^{-\gamma}. \qquad ...(3.4)$$

Similarly

$$\frac{d}{dt}\left[\frac{1 - \lambda t + R}{1 + \lambda t + R}\right]^{-\delta} = \lambda\delta\, R^{-1} \left(\frac{1 - \lambda t + R}{1 + \lambda t + R}\right)^{-\delta}. \qquad ...(3.5)$$

Now we write

$$(1 + \lambda t + R)^{-\alpha} (1 - \lambda t + R)^{-\beta}$$

$$= (1 - \lambda t + R)^{-\gamma} (1 + \lambda t + R)^{-\gamma} \left(\frac{1 - \lambda t + R}{1 + \lambda t + R}\right)^{-\delta} \qquad ...(3.6)$$

where

$$\gamma = \frac{\alpha + \beta}{2}, \qquad \delta = \frac{\beta - \alpha}{2}. \qquad ...(3.7)$$

By using (3.4), (3.5) and (3.6) we have

$$2^{\alpha+\beta} \frac{d}{dt}[t^\gamma (1 + \lambda t + R)^{-\alpha} (1 - \lambda t + R)^{-\beta}]$$

$$= 2^{\alpha+\beta} \frac{d}{dt}\left[t^\gamma (1 - \lambda t + R)^{-\gamma} (1 + \lambda t + R)^{-\gamma} \left(\frac{1 - \lambda t + R}{1 + \lambda t + R}\right)^{-\delta}\right]$$

$$= t^{\gamma-1} (\gamma + \lambda\delta t)\, 2^{\alpha+\beta}\, R^{-1} (1 - \lambda t + R)^{-\gamma} (1 + \lambda t + R)^{-\gamma}$$

$$\times \left[\frac{1 - \lambda t + R}{1 + \lambda t + R}\right]^{-\delta}$$

$$= t^{\gamma-1}(\gamma+\lambda\delta t)\, 2^{\alpha+\beta}\, R^{-1}(1+\lambda t+R)^{-\alpha}\,(1-\lambda t+R)^{-\beta}$$

$$= t^{\gamma-1}\,(\gamma+\lambda\delta t)\,\sum_{n=0}^{\infty} F_n(\alpha,\beta\,;x)\,t^n \quad \text{by (3.2).}$$

Thus

$$2^{\alpha+\beta}\frac{d}{dt}\,[t^{\gamma}\,(1+\lambda t+R)^{-\alpha}\,(1-\lambda t+R)^{-\beta}]$$

$$= \sum_{n=0}^{\infty}\,[\gamma\, F_n(\alpha,\beta\,;x)+\lambda\delta\, F_{n-1}(\alpha,\beta\,;x)]\,t^{\gamma+n-1}. \quad \text{...(3.8)}$$

Integrating both sides we have,

$$2^{\alpha+\beta}(1+\lambda t+R)^{-\alpha}\,(1-\lambda t+R)^{-\beta}$$

$$= \sum_{n=0}^{\infty}\frac{\gamma}{\gamma+n}\,F_n(\alpha,\beta\,;x)\,t^n+\sum_{n=0}^{\infty}\frac{\lambda\delta}{\gamma+n}\,F_{n-1}(\alpha,\beta\,;x)\,t^n \quad \text{...(3.9)}$$

provided $\gamma \neq 0$.

When $\gamma = 0$ we have on account of (3.7)

$$\left(\frac{1-\lambda t+R}{1+\lambda t+R}\right)^{-\beta} = 1+\sum_{n=1}^{\infty}\frac{\lambda\beta}{n}\,F_{n-1}(-\beta,\beta\,;x)\,t^n. \quad \text{...(3.10)}$$

Similarly when $\delta = 0$, we have

$$2^{2\beta}\,[(1+\lambda t+R)\,(1-\lambda t+R)]^{-\beta} = \sum_{n=0}^{\infty}\frac{\beta}{\beta+n}\,F_n(\beta,\beta\,;x)\,t^n \quad \text{...(3.11)}$$

provided β is not a negative integer.

4. Particular Cases

Jacobi Polynomials

If we put $-a = b = \lambda = 1$ in (3.9) and because of relationship (1.6) we obtain the generating function obtained by Carlitz (1961) in the form:

$$2^{\alpha+\beta}\,(1-t+\rho)^{-\alpha}\,(1+t+\rho)^{-\beta}$$

$$= \sum_{n=0}^{\infty}\,\{\gamma\, P_n^{(\beta,\alpha)}(x)+\delta\, P_{n-1}^{(\beta,\alpha)}(x)\}\,\frac{t^n}{\gamma+n}. \quad \text{...(4.1)}$$

where $\rho = (1+2xt+t^2)^{\frac{1}{2}}$ and $\gamma \neq 0$.

Laguerre Polynomials

Put $\lambda = 1$, $a = 0$, $\beta = b$ in (3.9) and in the limit $b \to \infty$,

$$R \to (1+t)\left[1 - \frac{2xt}{b(1+t)^2}\right]$$

$$\therefore \lim_{b\to\infty} \; 2^b \{2(1+t)\}^{-a} \left[2 - \frac{2xt}{b(1+t)}\right]^{-b}$$

$$= \lim_{b\to\infty} \sum_{n=0}^{\infty} \left[\frac{(a+b)/2}{\left(\frac{a+b}{2}\right)+n} F_n(a, b; x)\, t^n \right.$$

$$\left. + \frac{(b-a)/2}{\frac{a+b}{2}+n} F_{n-1}(a, b; x)\, t^n \right]$$

$$\therefore (1+t)^{-a} \exp\left(\frac{tx}{1+t}\right) = \sum_{n=0}^{\infty} [L_n^{(a)}(x) - L_{n-1}^{(a)}(x)](-t)^n.$$

Here we have used the result

$$\lim_{b\to\infty} \left(1 - \frac{t}{b}\right)^b = e^{-t}. \qquad \ldots(4.2)$$

Thus

$$(1-t)^{-a} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} [L_n^{(a)}(x) - L_{n-1}^{(a)}(x)]\, t^n. \qquad \ldots(4.3)$$

Since we have the recurrence relation for the Laguerre polynomials (Rainville 1976, p. 203),

$$L_n^{(a)}(x) = L_{n-1}^{(a)}(x) + L_n^{(a-1)}(x)$$

hence (4.3) becomes,

$$\frac{1}{(1-t)^{1+a}} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(a)}(x)\, t^n, \qquad \ldots(4.4)$$

which is the usual generating function for the Laguerre polynomials.

Hermite Polynomials

Put $\beta = \alpha$, $-a = b = \sqrt{\alpha}$, $(\alpha > 0)$ and since $\lambda \to 2/\sqrt{\alpha}$ in (3.9) and in the limit $\sqrt{\alpha} \to \infty$ we have $R \to 1 - \frac{2xt}{\alpha} + \frac{2t^2}{\alpha}$.

$$\therefore \lim_{\sqrt{\alpha}\to\infty} 2^{2\alpha}\left[(1+R)^2 - \frac{4t^2}{\alpha}\right]^{-\alpha} = \lim_{\sqrt{\alpha}\to\infty} \sum_{n=0}^{\infty} \frac{\alpha}{\alpha+n} F_n(\alpha, \alpha; x)\, t^n.$$

Thus on account of (1.8) and using (4.2) we get,

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n. \qquad \ldots(4.5)$$

This is the generating function for the Hermite polynomials. We can also obtain (4.5) from (3.11) on account of (1.8) and using (4.2).

Bessel polynomials

Put $-a = b = \lambda = 1$, $\alpha = r - \epsilon - 1$, $\beta = \epsilon - 1$, replace x by $\left(1 + \frac{2x\epsilon}{S}\right)$ and t by $\frac{sw}{2\epsilon}$ in (3.9), then we have

$$R \to \sqrt{1 - 2xw}.$$

Thus on account of (1.9) the left-hand side of (3.9) becomes:

$$\left[\frac{1}{2} + \frac{1}{2}\sqrt{1-2xw}\right]^{2-r}$$

$$\times \lim_{\epsilon\to\infty} \sum_{m=0}^{\infty} \left[\frac{(1+\epsilon-r)_m}{m!\,\epsilon^m} \left(\frac{sw/2}{1+\sqrt{1-\frac{sw}{\epsilon}\left(1+\frac{2x\epsilon}{s}\right)+\left(\frac{sw}{2\epsilon}\right)^2}}\right)^m\right]$$

$$\times \lim_{\epsilon\to\infty} \sum_{k=0}^{\infty} \left[\frac{(\epsilon-1)_k}{k!\,\epsilon^k} \left(\frac{sw/2}{1+\sqrt{1-\frac{sw}{\epsilon}\left(1+\frac{2x\epsilon}{s}\right)+\left(\frac{sw}{2\epsilon}\right)^2}}\right)^k\right]$$

$$= \left[\frac{1}{2} + \frac{1}{2}\sqrt{1-2xw}\right]^{2-r} \exp\left[\frac{s}{2x}\left(1-\sqrt{1-2xw}\right)\right].$$

And the right-hand side of (3.9) becomes,

$$\lim_{\epsilon\to\infty} \sum_{n=0}^{\infty} \frac{(r-2)}{(r-2+2n)} F_n\left(r-\epsilon-1,\ \epsilon-1\,;\,1+\frac{2x\epsilon}{s}\right)\left(\frac{sw}{2\epsilon}\right)^n$$

$$+\lim_{\epsilon\to\infty} \sum_{n=0}^{\infty} \frac{(2\epsilon-r)}{(r-2+2n)} F_{n-1}\left(r-\epsilon-1,\ \epsilon-1\,;\,1+\frac{2x\epsilon}{s}\right)\left(\frac{sw}{2\epsilon}\right)^n$$

This on account of (1.9) becomes,

$$\sum_{n=0}^{\infty}\left(\frac{r-2}{r-2+2n}\right) y_n\ (x\,;r,s)\left(\frac{sw}{2}\right)^n\frac{1}{n\,!}$$

$$+\sum_{n=0}^{\infty}\left(\frac{2}{r-2+2n}\right) y_{n-1}\ (x\,;r,s)\left(\frac{sw}{2\epsilon}\right)^n\frac{1}{(n-1)\,!}\,.$$

Thus when $r\neq 2$ we write

$$\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-2xw}\ \right]^{2-r}\exp\left[\frac{s}{2x}\left(1-\sqrt{1-2xw}\right)\right]$$

$$=\sum_{n=0}^{\infty}\left[\left(\frac{r-2}{2}\right)y_n\ (x\,;r,s)+ny_{n-1}(x\,;r,s)\ \right]\frac{(sw/2)^n}{n\,!\left(\frac{r-2+2n}{2}\right)}$$

...(4.6)

This result is not to be found in the literature.

Now when $r=2$, (4.6) becomes,

$$\exp\left[\frac{s}{2x}\left(1-\sqrt{1-2xw}\right)\right]=\sum_{n=0}^{\infty} y_{n-1}\,(x\,;2,s)\ \frac{(sw/2)^n}{n\,!} \qquad \text{...(4.7)}$$

Hence for $r=s=2$, we get the generating function for the simple Bessel polynomials (see Krall and Frink 1949, McBride 1971) in the form

$$\exp\left\{\frac{1-\sqrt{1-2xw}}{x}\right\}=\sum_{n=0}^{\infty}\frac{1}{n\,!}\,y_{n-1}\,(x)\,w^n.$$

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