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OSCILLATION AND NONOSCILLATION THEOREMS FOR SECOND ORDER QUASILINEAR FUNCTIONAL DIFFERENCE EQUATIONS

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In this paper, we have established necessary and sufficient conditions for the quasilinear functional difference equation

$$\Delta(|\Delta y(n)|)^{\alpha-1}\Delta y(n)\_ + f(n, y(\sigma(n))) = 0, \quad n \in N_0, \qquad \text{... (E)}$$

where $\alpha > 0$, to have various types of nonoscillatory solutions. Further we established some new oscillation conditions for the oscillation of all solutions of equation (*E*).

Key Words : Oscillation; Nonoscillation; Quasilinear; Functional Difference Equation

1. INTRODUCTION

Consider the difference equation

$$\Delta(|\Delta y(n)|^{\alpha-1}\Delta y(n))\_ + f(n, y(\sigma(n))) = 0, \qquad \text{... (E)}$$

where $n \in N_0 = \{n_0, n_0+1, n_0+2, \ldots\}$ (n_0 is a fixed nonnegative integer) and Δ is the forward difference operator defined by $\Delta y(n) = y(n+1) - y(n)$. Further we assume the following conditions without further mention :

(c_1) α is a positive constant;

(c_2) $\{\sigma(n)\}$ is a positive increasing sequence of integers such that $\lim_{n\to\infty} \sigma(n) = \infty$;

and (c_3) $f : N_0 \times R \to R$ is a continuous function, $u\, f(n, u) > 0$, for $u \neq 0$ and $f(n)$ is increasing for each fixed $n \in N_0$.

By a solution of (*E*), we mean a nontrivial real sequence $\{y(n)\}$ satisfying eq. (*E*) for all $n \geq n_0 - M$ where $M = \min_{n \in N_0} \{\sigma(n)\}$. A solution $\{y(n)\}$ of (*E*) is said to nonoscillatory if it is either eventually positive or eventually negative and oscillatory otherwise.

Note that the unperturbed equation $\Delta(|\Delta y(n)|^{\alpha-1}\Delta y(n)) = 0$, has the solutions $\{1, n\}$ and hence we can classify the possible nonoscillatory solutions of eq. (*E*) according to their asymptotic behavior as $n \to \infty$in the following manner :

(*I*) $\lim\limits_{n\to\infty} \dfrac{y(n)}{n} = \text{constant} \neq 0;$

(*II*) $\lim\limits_{n\to\infty} \dfrac{y(n)}{n} = 0, \qquad \lim\limits_{n\to\infty} |y(n)| = \infty;$

(*III*) $\lim\limits_{n\to\infty} y(n) = \text{constant} \neq 0.$

Solutions of the types (I), (II), (III) are called, respectively dominant, intermediate and subdominant solutions.

Recently, the oscillatory and asymptotic properties of the solutions of second order difference equations of the type (*E*) and/or related equations have been investigated by many authors, for example see [1, 2, 4-6, 8, 9, 12, 13] and the references cited therein. Following this trend in Section 2, we investigate the existence of these three types of solutions for eq. (*E*) showing that necessary and sufficient conditions can be obtained for the existence of dominant and subdominant solutions and in Section 3, we obtain some new criteria for oscillation of all solutions of eq. (*E*). Then, we have established that there exists a class of equation of the form (*E*) for which the oscillation solution can be completely characterized. The results in this paper have been motivated by that of in [3, 10, 11].

2. Existence of Nonoscillatory Solutions

In this we state the theorems for the existence of solutions of type (I), (II) and (III) for the eq. (*E*) without proof since the proofs of these theorems can be modelled as that of in [12]. Hence we omit the details.

Theorem 1 — *The difference equation* (E) *possesses a dominant solution if and only if there exists a constant* $c \neq 0$ *such that*

$$\sum_{n=n_0}^{\infty} |f(n, c\sigma(n))| < \infty. \qquad \dots (2.1)$$

Theorem 2 — *The difference equation (E) possesses a subdominant solution if and only if there exists a constant* $c \neq 0$ *such that*

$$\sum_{n=n_0}^{\infty} \left(\sum_{s=n}^{\infty} |f(s, c)| \right)^{1/\alpha} < \infty. \qquad \dots (2.2)$$

Theorem 3 — *Suppose that (2.2) holds for some* $c \neq 0$. *In addition, assume that*

$$\sum_{n=n_0}^{\infty}\left(\sum_{s=n}^{\infty}|f(s,c)|\right)^{1/\alpha}=\infty \qquad \text{... (2.3)}$$

for all $d \neq 0$ with $cd > 0$. Then the difference equation (E) has a nonoscillatory solution of type (II).

Example 1 — Consider the difference equation

$$\Delta\left(|\Delta y(n)|^{\alpha-1}\Delta y(n)\right)+q(n)\,|y(\sigma(n))|^{\beta-1}y(\sigma(n))=0, \qquad \text{... } (E_1)$$

where α and β are positive constants and $\{q(n)\}$ is a positive sequence. Clearly, all conditions of (E_1) are satisfied for this equation. It is easy to see that the conditions (2.1) and (2.2) reduce, respectively to

$$\sum_{n=n_0}^{\infty}(\sigma(n))^{\beta}q(n)<\infty \qquad \text{... (2.4)}$$

and

$$\sum_{n=n_0}^{\infty}\left(\sum_{s=n}^{\infty}q(s)\right)^{1/\alpha}<\infty. \qquad \text{... (2.5)}$$

It follows that :

(*i*) eq. (E_1) has a dominant solution if and only if (2.4) holds;

(*ii*) eq. (E_1) has subdominant solution if and only if (2.5) holds; and

(*iii*) eq. (E_1) has a intermediate solution if

$$\sum_{n=n_0}^{\infty}(\sigma(n))^{\beta}q(n)<\infty \quad \text{and} \quad \sum_{n=n_0}^{\infty}\left(\sum_{s=n}^{\infty}q(s)\right)^{1/\alpha}=\infty. \qquad \text{... (2.6)}$$

In particular suppose that $q(n)=n^{-\delta}, n\geq 1$ where $\delta>0$ is a constant. As is easily verified.

(*i*) If $\sigma(n)=an+b$, a is a positive integer and b any integer, then
(2.4) holds if and only if $\delta>1+\beta$; (2.5) holds if and only if $\delta>\alpha+1$;
(2.6) holds if and only if $\alpha\geq\beta$ and $1+\beta<\delta\leq\alpha+1$;.

(*ii*) If $\sigma(n)=n^{\gamma}$, γ is a positive integer, then
(2.4) holds if and only if $\delta>1+\gamma\beta$; (2.5) holds if and only if $\delta>1+\alpha$;
(2.6) holds if and only if $\alpha>\beta\gamma$ and $1+\beta\gamma<\delta\leq 1+\alpha$.

3. Oscillation Results

In this section we study the oscillatory behaviour of solutions of the eqs. (E). We begin with the following lemmas :

Lemma 4 — Let $\{y(n)\}$ be a sequence such that $y(n) > 0$, $\Delta y(n) > 0$ and nonincreasing for $n \geq N_1 \in N_0$. Let $\{\sigma(n)\}$ be a sequence of positive integers such that $\lim_{n \to \infty} \sigma(n) = \infty$. Then, for every $l \in (0, 1)$, there exists an integer $N \geq N_1$ such that

$$y(\sigma(n)) \geq l \frac{\sigma_*(n)}{n} y(n), \quad n \geq N, \qquad \text{... (3.1)}$$

where $$\sigma_*(n) = \min\{n, \sigma(n)\}. \qquad \text{... (3.2)}$$

PROOF : Since $y(n)$ is increasing and $\{\Delta y(n)\}$ is nonincreasing, we have

$$y(\sigma(n)) \geq y(\sigma_*(n)), \quad n \geq N_1 \qquad \text{... (3.3)}$$

and

$$y(n) - y(\sigma_*(n)) = \sum_{s=\sigma_*(n)}^{n-1} \Delta y(s) \leq \Delta y(\sigma_*(n))(n - \sigma_*(n)), \quad n \geq \sigma_*(n) \geq N_1.$$

It follows that

$$\frac{y(n)}{y(\sigma(n))} \leq 1 + \frac{\Delta y(\sigma_*(n))}{y(\sigma_*(n))}(n - \sigma_*(n)), \quad n \geq \sigma_*(n)), \quad n \geq \sigma_*(n) \geq N_1. \qquad \text{... (3.4)}$$

On the other hand,

$$y(\sigma_*(n)) - y(N_1) = \sum_{s=N_1}^{\sigma_*(n)-1} \Delta y(s) \geq \Delta y(\sigma_*(n))(\sigma_*(n) - N_1)$$

which implies that for each $l \in (0, 1)$ there exists an integer $N \geq N_1$ such that

$$\frac{y(\sigma_*(n))}{\Delta y(\sigma_*(n))} \geq l(\sigma_*(n)), \quad n \geq N. \qquad \text{... (3.5)}$$

Combining (3.4) and (3.5) we have

$$\frac{y(n)}{y(\sigma(n))} \leq 1 + \frac{(n - \sigma_*(n))}{l(\sigma_*(n))} = \frac{(l-1)\sigma_*(n) + n}{l(\sigma_*(n))} \leq \frac{n}{l(\sigma_*(n))}, \quad n \geq N$$

which completes the proof.

Lemma 5 — Assume $f(n, u)$ satisfies condition (c_3). Let α and $\{\sigma(n)\}$ be as in eq. (E). If the difference inequality

$$\Delta (| \Delta x(n) |^{\alpha - 1} \Delta x(n)) + f(n, x(\sigma(n))) \leq 0 \qquad \text{... (3.6)}$$

has an eventually positive solution, then so does the difference equation

$$\Delta (| \Delta y(n) |^{\alpha - 1} \Delta y(n)) + f(n, y(\sigma(n))) = 0. \qquad \text{... (3.7)}$$

PROOF : Let $\{x(n)\}$ be an eventually positive solution of (3.6). Let $N \in N_0$ be such that $x(n) > 0$ and $x(\sigma(n)) > 0$ for all $n \geq N$. From (3.6), one can easily see that $\Delta x(n) > 0$ and decreasing for $n \geq N$. Summing (3.6) from n to ∞, we obtain,

$$(\Delta x(n))^{\alpha} \geq \omega + \sum_{s=n}^{\infty} f(s, x(\sigma(s))), \quad n \geq N,$$

where $\omega = \lim_{n \to \infty} (\Delta x(n))^{\alpha} \geq 0$, that is

$$\Delta x(n) \geq \left(\omega + \sum_{s=n}^{\infty} f(s, x(\sigma(s))\_ \right)^{1/\alpha}, \quad n \geq N.$$

Summing the last inequality from N to $n - 1$ yields

$$x(n) \geq x(N) + \sum_{s=N}^{n-1} \left(\omega + \sum_{r=s}^{\infty} f(r, x(\sigma(r))) \right)^{1/\alpha}, \quad n \geq N.$$

Let $N_* = \min \left\{ N, \inf_{n \geq N} \sigma(n) \right\} \geq n_0$. Consider the Banach space B_{N_*} of all real sequences $y \mid = \{y(n)\}_{n \geq N_*}$ with the supremum norm $\| y \| = \sup_{n \geq N_*} \left\{ \frac{y(n)}{\rho(n)} \right\}$ with $\rho(n) = (n - N)_+ = (n - N)$ if $n \geq N$ and is zero otherwise. We define a set S as

$$S = \{ y \in B_{N_*} : 0 \leq y(n) \leq x(n), n \geq N + 1 \ \text{ and } \ y(n) = x(N) \ \text{ for } \ N_* \leq n \leq N \}.$$

Clearly, S is a closed, bounded and convex subset of B_{N_*}. We Define a partial order on B_{N_*} in the usual way. That is, for any $x = \{x(n)\}$, $y = \{y(n)\} \in B_{N_*}$, $x(n) = y(n)$ for all $n >> 1$, we will consider such sequences to be the same. Thus for every subset A of S, both inf A and sup A exist and belong to S. Now define an operator $T : S \to B_{N_*}$ as

$$(Ty)(n) = \begin{cases} x(N) + \sum_{s=N}^{n-1} \left(\omega + \sum_{r=s}^{\infty} f(r, y(\sigma(r))) \right)^{1/\alpha}, & n \geq N+1, \\ x(N), \quad N_* \leq n \leq N. & \end{cases}$$

From the hypothesis T is increasing. If $y \in S$, then we see that $(Ty)(n) \leq x(n)$ for all $n \geq N+1$ and $(Ty)(n) \geq 0$. Thus $TS \subseteq S$. Therefore by the Knaster-Taraski fixed point theorem[7], T has a fixed point $y \in S$ such that $(Ty)(n) = y(n)$ and satisfies the equation

$$y(n) = x(N) + \sum_{s=N}^{n-1} \left(\omega + \sum_{r=s}^{\infty} f(r, y(\sigma(r))) \right)^{1/\alpha}, \quad n \geq N.$$

This clearly shows that $\{y(n)\}$ is a positive solution of eq. (3.7).

First we present a criterion for the oscillation of all bounded solutions of eq. (E).

Theorem 6 — *All bounded solutions of equation (E) are oscillatory if*

$$\sum_{n=n_0}^{\infty} \left(\sum_{s=n}^{\infty} \left| f\left(s, c\frac{\sigma_*(s)}{s} \right) \right| \right)^{1/\alpha} = \infty \qquad \text{... (3.8)}$$

for all $c \neq 0$.

PROOF : Suppose to the contrary that eq. (E) has a bounded nonoscillatory solution $\{y(n)\}$. Without loss of generality we may assume that $\{y(n)\}$ is eventually positive. Then $\Delta y(n)) > 0$ and nonincreasing and hence applying Lemma 4, we see that for every $l \in (0, 1)$ $\{y(n)\}$ satisfies (3.1) provided N is sufficiently large. From eq. (E) and (3.1) we obtain the following inequality

$$\Delta(\,|\Delta y(n)|)^{\alpha-1} \Delta y(n)) + f\left(n, \frac{l\,\sigma_*(n)}{n} y(n) \right) \leq 0, \quad n \geq N.$$

Now from Lemma 5, there exists a sequence $\{z(n)\}$ for $n \geq N$ such that $0 < z(n) \leq y(n)$ and that

$$\Delta\left((|\Delta z(n)|)^{\alpha-1} \Delta z(n))_{-} + f\left(n, \frac{l\sigma_*(n)}{n} z(n) \right) \right) = 0, \quad n \geq N. \qquad \text{... (3.9)}$$

Thus $\{z(n)\}$ is a bounded nonoscillatory solution (subdominant solution) of (3.9), and so application of Theorem 2 shows that

$$\sum_{n=N}^{\infty} \left(\sum_{s=n}^{\infty} f\left(s, \frac{k\sigma_*(s)}{s} \right) \right)^{1/\alpha} < \infty.$$

This contradicts (3.8) and the proof is complete.

Corollary 1 — Suppose that $\liminf_{n\to\infty} \frac{\sigma(n)}{n} > 0$. Then all bounded solution of eq. (*E*) are oscillatory if and only if

$$\sum_{n=n_0}^{\infty}\left(\sum_{s=n}^{\infty} |f(s,c)|\right)^{1/\alpha} = \infty \qquad \text{... (3.10)}$$

for all $c \neq 0$.

Theorem 7 — *Assume that there exsts a continuous function $\phi(u)$ on R which is nondecreasing and satisfies $u\,\phi(u) > 0$, for $u \neq 0$,*

$$\int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty \qquad \text{... (3.11)}$$

and

$$\liminf_{u\to\infty} \frac{|f(n,uv)|}{|\phi(u)|^{\alpha}} \geq k_1 |f(n,v)|, \quad n \geq N_0, \qquad \text{... (3.12)}$$

for some constant $k_1 > 0$ and all v with $|v| \leq 1$. If condition (3.8) holds, then all solutions of eq. (E) are oscillatory.

PROOF : Suppose to the contrary that eq. (*E*) has a nonoscillatory solution $\{y(n)\}$ which is evetually positive. As in the proof of Theorem 6, we obtain (3.9) with l replaced by some δ such that $0 < \delta < 1$. From Theorem 6, it follows that $\{z(n)\}$ cannot be bounded for $n \geq N$, that is $\lim_{n\to\infty} z(n) = \infty$. From (3.9) we obtain

$$\Delta z(n) \geq \left(\sum_{s=n}^{\infty} f\left(\sum_{s=n}^{\infty} f\left(s, \frac{\delta\sigma_*(s)}{s} z(s)\right)\right)\right)^{1/\alpha}, \quad n \geq N \qquad \text{... (3.13)}$$

and

$$\frac{\Delta z(n)}{\phi(\delta z(n))} \geq \frac{1}{\phi(\delta z(n))}\left(\sum_{s=n}^{\infty} f\left(s, \frac{\delta\sigma_*(s)}{s} z(s)\right)\right)^{1/\alpha} \geq \left(\sum_{s=n}^{\infty} \frac{f\left(s, \frac{\delta\sigma_*(s)}{s} z(s)\right)}{(\phi(\delta z(s))^{\alpha}}\right)^{1/\alpha} \qquad \text{... (3.14)}$$

Using (3.12) in (3.14), we obtain

$$\frac{\Delta z(n)}{\phi(\delta z(n))} \geq \left(\frac{k_1}{2}\right)^{1/\alpha} \left(\sum_{s=n}^{\infty} f\left(s, \frac{\sigma_*(s)}{s}\right)\right)^{1/\alpha} \quad n \geq N. \qquad \ldots (3.15)$$

Setting $\delta z(n-1) \leq u \leq \delta z(n)$ implies that $\frac{1}{\phi(u)} \geq \frac{1}{\phi(\delta z(n))}$ and summing the last inequality from N to $n-1$, we obtain

$$\frac{1}{\delta} \int_{\delta z(N)}^{\delta z(n)} \frac{du}{\phi(u)} \geq \left(\frac{k_1}{2}\right)^{1/\alpha} \sum_{s=N}^{n-1} \left(\sum_{r=s}^{\infty} f\left(r, \frac{\sigma_*(r)}{r}\right)\right)^{1/\alpha} \quad n \geq N. \qquad \ldots (3.16)$$

Letting $n \to \infty$ in (3.16) and then using (3.11), we have

$$\sum_{s=N}^{\infty} \left(\sum_{r=s}^{\infty} f\left(r, \frac{\sigma_*(r)}{r}\right)\right)^{1/\alpha} < \infty,$$

which contradicts (3.8). This proves the theorem.

Corollary 2 — Assume that $\liminf_{n \to \infty} \frac{\sigma(n)}{n} > 0$. Suppose there exists, a function $\phi(u)$ with the properties as stated in Theorem 7. Then, all solutions of eq. (E) are oscillatory if and only if (3.10) holds.

Theorem 8 — *Assume that there exists a continuous function $\Psi(u)$ on R which is nondecreasing and satisfies $u\,\Psi(u) > 0$, for $u \neq 0$,*

$$\liminf_{u \to \infty} \frac{|f(n, uv)|}{|\Psi(u)|} \geq k_2 |f(n, v)| \qquad \ldots (3.17)$$

for some $k_2 > 0$ and all v with $|v| \geq 1$. If

$$\int_0^{\pm\beta} \frac{du}{\Psi(u^{1/\alpha})} < \infty \quad \text{for all} \quad \beta > 0 \qquad \ldots (3.18)$$

and

$$\sum_{n=n_0}^{\infty} |f(n, c\sigma_*(n))| = \infty \qquad \ldots (3.19)$$

for all $c \neq 0$, then all solutions of eq. (E) are oscillatory.

PROOF : Suppose that $\{y(n)\}$ is an eventually positive solution of eq. (E). Since (3.19) implies

$$\sum_{n=n_0}^{\infty} |f(n, c\sigma(n))| = \infty$$

for all $c \neq 0$, eq. (E) cannot possess dominant solution by Theorem 1, so that $\{y(n)\}$ is either subdominant or intermediate.

$$\lim_{n \to \infty} \Delta y(n) = \lim_{n \to \infty} \frac{y(n)}{n} = 0.$$

Since $\Delta y(n) \geq 0$ and decreasing, we have

$$y(n) - y(n_0) = \sum_{s=n_0}^{n-1} \Delta y(s) \geq \Delta y(n)\,(n - n_0), \quad n \geq n_0,$$

which implies that for a fixed $\delta_1, 0 < \delta_1 < 1$ and for $N > n_0$ sufficiently large

$$y(n) \geq \delta_1\, n\, \Delta y(n), \quad n \geq N.$$

This implies that

$$y(\sigma(n)) \geq y(\sigma_*(n)) \geq \delta_1\, \sigma_*(n)\, \Delta y(\sigma_*(n)) \geq \delta_1\, \sigma_*(n)\, \Delta y(n), \quad n \geq N_1 \geq N,$$

and hence

$$\frac{y(\sigma(n))}{\Delta y(n)} \geq \delta_1\, \sigma_*(n), \quad n \geq N_1. \qquad \text{... (3.20)}$$

Note that

$$\Delta\left(\sum_{s=n}^{\infty} \frac{\Delta((\Delta y(s))^{\alpha})}{\Psi(\Delta y(s)}\right) = \frac{f(n, y(\sigma(n)))}{\Psi(\Delta y(n))} \geq \frac{f(n, \delta_1\, \sigma_*(n)\Delta y(n))}{\Psi(\Delta(y(n))} \; n \geq N_1, \qquad \text{... (3.21)}$$

where (3.20) has been used. In view of (3.17), $N_2 \geq N_1$ can be chosen so large that $\delta_1\, \sigma_*(n) \geq 1$ for $n \geq N_2$ and that

$$\frac{f(n, \delta_1 \sigma_*(n)\, \Delta y(n))}{\Psi(\Delta y(n))} \geq \frac{k_2}{2} f(n, \delta_1\, \sigma_*(n)), \qquad n \geq N_2,$$

which combined with (3.21) yields

$$\frac{k_2}{2}\sum_{s=N_2}^{n-1} f(s, \delta_1\, \sigma_*(s)) \leq -\sum_{s=N_2}^{n-1} \frac{\Delta((\Delta y(s))^{\alpha})}{\Psi(\Delta y(s))} \leq \int_0^{(\Delta y_{N_2})^{\alpha}} \frac{du}{\Psi(u^{1/\alpha})}. \quad \text{.. (3.22)}$$

Letting $n \to \infty$ in (3.22) and using (3.18), we obtain $\sum_{s=N_2}^{\infty} f(s, \delta_1\, \sigma_*(s)) < \infty$, which contradicts (3.19). This completes the proof of the theorem.

Corollary 3 — Assume that there exists a function $\Psi(u)$ with the property as in Theorem 8.

(*i*) Suppose that $\sigma(n) \leq n$ for all $n \in N_0$. Then all solutions of eq. (*E*) are oscillatory if and only if

$$\sum_{n=n_0}^{\infty} |f(n, c\sigma(n))| = \infty$$

for all $c \neq 0$.

(*ii*) Suppose that

$$\liminf_{n\to\infty} \frac{\sigma(n)}{n} > 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{\sigma(n)}{n} < \infty. \quad \text{... (3.23)}$$

Then all solutions of eq. (*E*) are oscillatory if and only if

$$\sum_{n=n_0}^{\infty} |f(n, cn)| = \infty \quad \text{... (3.24)}$$

for all $c \neq 0$.

Example 2 — Consider the difference equation

$$\Delta(|\Delta y(n)|^{\alpha-1} \Delta y(n)) + q(n) |y(\sigma(n)|^{\beta-1} y(\sigma(n)) = 0. \quad \text{... (E2)}$$

(*I*) Let $\beta > \alpha$. All solutions of eq. (E_2) are oscillatory if

$$\sum_{n=n_0}^{\infty} \left(\sum_{s=n}^{\infty} q(s) \left(\frac{\sigma_*(s)}{s} \right)^{\beta} \right)^{1/\alpha} = \infty.$$

Suppose in addition that $\{\sigma(n)\}$ satisfies

$$\liminf_{n\to\infty} \frac{\sigma(n)}{n} > 0.$$

Then, all solutions of eq. (E_2) are oscillatory if and only if

$$\sum_{n=n_0}^{\infty} \left(\sum_{s=n}^{\infty} q(s) \right)^{1/\alpha} = \infty.$$

(II) Let $\beta < \alpha$. All solutions of eq. (E_2) are oscillatory if

$$\sum_{n=n_0}^{\infty} q(n)\,(\sigma_*(n))^{\beta} = \infty.$$

Suppose in addition that $\sigma(n) \leq n$ for $n \in N_0$. Then all solutions of eq. (E_2) are oscillatory if and only if

$$\sum_{n=n_0}^{\infty} q(n)\,(\sigma(n))^{\beta} = \infty.$$

On the other hand, in case $\{\sigma(n)\}$ satisfies (3.23), a necessary and sufficient condition for all solutions of eq. (E_2) is that

$$\sum_{n=n_0}^{\infty} n^{\beta} q(n) = \infty.$$

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