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NORMALITY CRITERIA OF FAMILIES OF MEROMORPHIC FUNCTIONS

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In this paper, we give a new approach to establish normality criteria, and improve previous results due to Pang, Schwick, Xu and Hua.

Key Words : Meromorphic Function; Multiple Pole; Normal Family

1. INTRODUCTION

Let G be a domain in C, and $\mathcal{F}$ be a family of meromorphic functions defined in G. $\mathcal{F}$ is said to be normal in G, in the sense of Montel, if for any sequence $f_n \in \mathcal{F}$ there exists a subsequence f_{n_j}, such that f_{n_j} spherically converges, locally and uniformly in G, to a meromorphic function or ∞.

In 1995, Chen and Fang[2] completely proved a Hayman's conjecture for normal families of meromorphic functions. In [2], they also proposed the following conjecture :

Conjecture — Let $\mathcal{F}$ be a family of meromorphic functions in a domain G. If every function $f \in \mathcal{F}$ satisfies $f^{(k)} - af^n \neq b$, where $a(\neq 0)$, b are two finite complex numbers, and $n \geq k+2$, then $\mathcal{F}$ is normal in G.

In fact, Pang[7] and Schwick[8] proved the following result :

***Theorem** A — Let $\mathcal{F}$ be a family of meromorphic functions in a domain G, $a_i(z)$ $(i = 1, ..., k-1)$ be holomorphic functions in G, and $a, b \in \mathbf{C}$ with $a \neq 0$, and let $n, k \in \mathbf{N}$ with $n \geq k+4$. If every function $f \in \mathcal{F}$ satisfies*

$$f^{(k)}(z) + \sum_{i=1}^{k-1} a_i(z) f^{(i)}(z) - af^n(z) \neq b, \qquad \text{... (1)}$$

then $\mathcal{F}$ is normal in G.

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Recently, the first author[9] proved :

Theorem B — *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain* G, $a(\neq 0), b \in \mathbf{C}$. *If* $n \geq k+2$ *and for every* $f \in \mathcal{F}$ *satisfies*

1) $f^{(k)} - af^n \neq b$, ... (2)

2) *f has only multiple poles,*

then $\mathcal{F}$ *is normal in* G.

As we knew, most of normality criteria are based on such a condition as (1) or (2). It is natural to ask : What can we said about the normality if (1) or (2) does not hold ?

In this paper, we shall solve this problem. Our results are as follows :

Theorem 1 — *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain* G, $a(z)\ (\neq 0)$, $b(z)$, *and* $a_i(z)\ (i = 1, \ldots, k-1)$ *be holomorphic functions in* G, *and let* $n, k \in N$ *with* $n \geq k+4$. *For each* $f \in \mathcal{F}$, *we define*

$$E(f) = \left\{ z : f^{(k)}(z) + \sum_{i=1}^{k-1} a_i(z) f^{(i)}(z) - a(z) f^n(z) = b(z),\ \ z \in G \right\}.$$

If there exists a constant $M > 0$ *such that, for every* $f \in \mathcal{F}$ *and* $z \in E(f)$,

$$|f^{(i)}(z)| \leq M, \qquad (i = 1, \ldots, k) \qquad \ldots (3)$$

then $\mathcal{F}$ *is normal in* G.

Theorem 2 — *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain* G, $a(z)\ (\neq 0)$, $b(z)$, *and* $a_i(z)\ (i = 1, \ldots, k-1)$ *be holomorphic functions in* G, *and let* $n, k \in N$ *with* $k \geq 2$, $n \geq k+2$. *Suppose that each function in* $\mathcal{F}$ *has only multiple poles. For each* $f \in \mathcal{F}$, *we define*

$$E(f) = \left\{ z : f^{(k)}(z) + \sum_{i=1}^{k-1} a_i(z) f^{(i)}(z) - a(z) f^n(z) = b(z),\ \ z \in G \right\}$$

If there exists a constant $M > 0$ *such that, for every* $f \in \mathcal{F}$ *and* $z \in E(f)$,

$$|f^{(i)}(z)| \leq M, \quad (i = 1, \ldots, k) \qquad \ldots (3)$$

then $\mathcal{F}$ *is normal in* G.

Remark : If $E(f)$ is empty, then (3) holds naturally. So, Theorem 1 and Theorem 2 include Theorem A and Theorem B respectively.

The second part of this paper is concerning with a result of Hua[6].

Theorem C[6] — *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain* G. *Suppose that there exists a constant* $M > 0$ *such that, for each* $f \in \mathcal{F}$,

$$|f(z)| \geq M$$

whenever $z \in E(f) = \{z : (f^n)^{(k)} = 1\}$. *If* $n \geq k+2$, *then* $\mathcal{F}$ *is normal in* G; *if* $n = k+1$, *then* $\mathcal{F}$ *is* *normal in G under an additional condition that f has only poles of order at least 5 for every* $f \in \mathcal{F}$.

We shall prove that the additional condition in Theorem C for the case $n = k + 1$ can be omitted.

***Theorem* 3** — *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain G. Suppose that there exists a constant* $M > 0$ *such that, for each* $f \in \mathcal{F}$,

$$|f(z)| \geq M$$

whenever $z \in E(f) = \{z : (f^n)^{(k)} = 1\}$. *If* $n \geq k+1$, *then* $\mathcal{F}$ *is normal in G.*

Remark 2 : Theorem 3 does not hold for $n = k$.

Example : Let $\mathcal{F} = \{iz : i = 2, 3, \ldots\}$, $G = \{|z| < \frac{1}{4}\}$, $n = k = 1$. $E(f) = \{z : (iz)' = 1\}$ $= \phi$. However, by Marty's criterion, $\mathcal{F}$ is not normal at 0.

Remark 3 : This result is a generalization of the problem No. 5.12 in Hayman[4].

In this paper, we use the standard notations of value distribution theory[5].

2. Lemmas

To prove our results, we need some lemmas.

Lemma 1 — Let f be a nonconstant meromorphic function on the complex plane which has only poles of order at least l, a be a nonzero finite complex number, $n, k, l \in N$ with $n \geq k+2$. Suppose that f is not a polynomial of degree less than k. Then

$$\left(n - 1 - \frac{k}{l}\right)T(r,f) \leq \frac{1}{l}N(r,f) + N(r, 1/f) + \overline{N}\left(r, \frac{1}{f^{(k)} - af^n}\right) + S(r,f),$$

where $S(r,f) = o(T(r,f))$, as $r \to \infty$, possibly outside a set with finite linear measure.

PROOF : It is easy to see that $f^{(k)} - af^n \not\equiv 0$. Let

$$\psi = \frac{f^{(k)}}{af^n}.$$

Thus from

$$f^n = \frac{f^{(k)}}{a\psi},$$

we deduce that

$$nm(r,f) \leq m\left(r, \frac{1}{\psi}\right) + m(r,f) + m\left(r, \frac{f^{(k)}}{f}\right) + O(1),$$

i.e.,

$$(n-1)\,m(r,f) \le m\left(r,\frac{1}{\psi}\right) + S(r,f). \qquad \ldots (4)$$

Obviously, a zero of ψ is either a pole of f or a zero of $f^{(k)}$, and a pole of f with order p must be a zero of ψ with order $(n-1)p-k\,(>0)$. Hence, if we denote by $N_0(r)$ the counting function of zeros of both ψ and $f^{(k)}$, we have

$$\overline{N}\left(r,\frac{1}{\psi}\right) = \overline{N}(r,f) + \overline{N}_0(r), \qquad \ldots (5)$$

and

$$N\left(r,\frac{1}{\psi}\right) = (n-1)\,N(r,f) - k\,\overline{N}(r,f) + N_0(r),$$

since each pole of f has order at least l, we have $\overline{N}(r,f) \le \frac{1}{l}N(r,f)$, thus

$$\left(n-1-\frac{k}{l}\right)N(r,f) \le N\left(r,\frac{1}{\psi}\right) - N_0(r), \qquad \ldots (6)$$

Combining (4), (6) and using Nevanlinna's second fundamental theorem, we obtain

$$\left(n-1-\frac{k}{l}\right)T(r,f) \le T\left(r,\frac{1}{\psi}\right) - N_0(r) + S(r,f)$$

$$\le \overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}(r,\psi) + \overline{N}\left(r,\frac{1}{\psi-1}\right) - N_0(r) + S(r,f). \qquad \ldots (7)$$

Since $\psi - 1 = (f^{(k)} - af^n)/(af^n)$, the pole of f cannot be a zero of $\psi - 1$, thus

$$\overline{N}\left(r,\frac{1}{\psi-1}\right) \le \overline{N}\left(r,\frac{1}{f^{(k)}-af^n}\right). \qquad \ldots (8)$$

It is easy to see that

$$\overline{N}(r,\psi) \le N(r,1/f). \qquad \ldots (9)$$

From the discussions above, we get

$$\left(n-1-\frac{k}{l}\right)T(r,f) \le \frac{1}{l}N(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-af^n}\right) + S(r,f).$$

This completes the proof of Lemma 1.

Lemma 2[1] — If f is a transcendental meromorphic function and $m > l$ are positive integers, then $(f^m)^{(l)}$ assumes every finite non-zero complex value infinitely often.

The following results are the improvements of the well-known Zalcman's principle.

Lemma 3[7] — Let $\mathcal{F}$ be a family of meromorphic functions. If $\mathcal{F}$ is not normal at a point z_0, then for $-1 < \alpha < 1$, there exists a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$, such that $\rho_j^{\alpha} f_j(z_j + \rho_j \zeta)$ spherically and uniformly converges to a non-constant meromorphic function on any disk $|\zeta| \le R$.

Lemma 4[3] — Let $\mathcal{F}$ be family of meromorphic functions with the property that every function $f \in \mathcal{F}$ has only zeros of order at least k. If $\mathcal{F}$ is not normal at a point z_0, then for $0 \le \alpha < k$, there exists a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$, such that $\rho_j^{-\alpha} f_j(z_j + \rho_j \zeta)$ spherically and uniformly converges to a non-constant meromorphic function on any disk $|\zeta| \le R$.

3. Proof of Theorems

Proof of Theorem 1 : Let $z_0 \in G$ and take $\alpha = \frac{k}{n-1}$, obviously $0 < \alpha < 1$. Assume that $\mathcal{F}$ is not normal at z_0. Then, by Lemma 3, there exists a sequence $g_j(\zeta) = \rho_j^{\frac{k}{n-1}} f_j(z_j + \rho_j \zeta)$ with $\rho_j \to 0$ and $z_j \to z_0$, which converges to a non-constant meromorphic function $g(\zeta)$ spherically and locally uniformly in $\boldsymbol{C}$. By Lemma 1 and the assumption of the theorem that $n \ge k + 4$, we know that there exists a point ζ_0 such that

$$g^{(k)}(\zeta_0) - a(z_0)\, g^n(\zeta_0) = 0, \qquad \ldots (10)$$

From (10) we have

$$g(\zeta_0) \ne \infty. \qquad \ldots (11)$$

Now for arbitrary small $\varepsilon > 0$, there exists $m_0 > 0$, such that, for $m > m_0$ and $|\zeta - \zeta_0| < \frac{1}{m}$,

$$|g^{(k)}(\zeta) - a(z_0)\, g^n(\zeta)| < \varepsilon.$$

Since $\rho_j \to 0$, thus near ζ_0, the function $g^{(k)}(\zeta) - a(z_0)\, g^n(\zeta)$ is the uniform limit of

$$g_j^{(k)}(\zeta) + \sum_{i=1}^{k-1} \rho_j^{k-1} a_i(z_j + \rho_j \zeta)\, g_j^{(i)}(\zeta) - a(z_j + \rho_j \zeta)\, g_j^{n}(\zeta) - \rho_j^{\frac{nk}{n-1}} b(z_j + \rho_j \zeta)$$

$$= \rho_j^{\frac{nk}{n-1}} [f_j^{(k)}(z_j+\rho_j\zeta) + \sum_{i=1}^{k-1} a_i(z_j+\rho_j\zeta) f_j^{(i)}(z_j+\rho_j\zeta)$$

$$- a(z_j+\rho_j\zeta) f_j^n(z_j+\rho_j\zeta) - b(z_j+\rho_j\zeta)].$$

By Rouche's theorem, there exists a point ζ_j with $|\zeta_j - \zeta_0| < \frac{1}{m}$ and

$$f_j^{(k)}(z_j+\rho_j\zeta_j) + \sum_{i=1}^{k-1} a_i(z_j+\rho_j\zeta_j) f_j^{(i)}(z_j+\rho_j\zeta_j) - a(z_j+\rho_j\zeta_j) f_j^n(z_j+\rho_j\zeta_j) = b(z_j+\rho_j\zeta_j),$$

then from our hypothesis we have

$$|f_j^{(i)}(z_j+\rho_j\zeta_j)| \le M, \qquad (i = 1, \ldots, k),$$

and so

$$|g_j^{(i)}(\zeta_j)| \le \rho_j^{\frac{k}{n-!}+1} M, \qquad (i = 1, \ldots, k). \qquad \ldots (12)$$

Without loss of generality, we may suppose that $\zeta_j \to \zeta_0'$, thus $|\zeta_0' - \zeta_0| < \frac{1}{m}$. Let $m \to \infty$, we have $\zeta_0' - \zeta_0$.

This and (12) give

$$g^{(i)}(\zeta_0) = 0, \quad (i, = 1, \ldots, k).$$

Combining this with (10) we get $g(\zeta_0) = 0$. Therefore, ζ_0 is a zero of $g(\zeta)$ with order not less than $k + 1$. We thus have

$$\overline{N}\left(r, \frac{1}{g^{(k)}(\zeta) - a(z_0)\, g^n(\zeta)}\right) \le \frac{1}{k+1} N\left(r, \frac{1}{g(\zeta)}\right), \qquad \ldots (13)$$

then, by Lemma 1 we have

$$(n-1-k)\, T(r, g) \le N(r, g) + N(r, 1/g) + \frac{1}{k+1} N(r, 1/g) + S(r, g),$$

that is

$$\left(n - k - 3 - \frac{1}{k+1}\right) T(r, g) \le S(r, g).$$

We arrive at a contradiction since $n \ge k+4$. This completes the proof of Theorem 1.

PROOF OF THEOREM 2 : Suppose on the contrary that $\mathcal{F}$ is not normal at $z_0 \in G$. Take $\alpha = \frac{k}{n-1}$, obviously $0 < \alpha < l$. Then, by applying Lemma 4 to $*G = \left\{ \frac{1}{f} : f \in \mathcal{F} \right\}$, there exists a sequence

$$g_j(\zeta) = \rho_j^{\frac{k}{n-1}} f_j(z_j + \rho_j \zeta)$$

such that $g_j(\zeta)$ spherically and locally uniformly convergent to a non-constant meromorphic function $g(\zeta)$.

Similarly as in the proof of Theorem 1, we have (10), (11), (12) and (13). Thus, by Lemma 1, we get

$$\left(n - 2 - \frac{k+1}{l} - \frac{1}{k+1} \right) T(r, g) \leq S(r, g).$$

However, according to the assumption we have

$$n - 2 - \frac{k+1}{l} - \frac{1}{k+1} \geq \frac{k-1}{2} - \frac{1}{k+1} > 0,$$

which implies that g is a constant, a contradiction. This proves Theorem 2.

PROOF OF THEOREM 3 : Let $z_0 \in G$ and take $\alpha = -\frac{k}{n}$, obviously $-1 < \alpha < 0$. If $\mathcal{F}$ is not normal at z_0, then by Lemma 3, there exists a sequence $g_j(\zeta) = \rho_j^{-\frac{k}{n}} f_j(z_j + \rho_j \zeta)$, a positive sequence $\rho_j \to 0$ and a sequence $z_j \to z_0$, such that $g_j(\zeta)$ converges to a non-constant meromorphic function $g(\zeta)$ spherically and locally uniformly in C. Since $n > k$, $(g^n)^{(k)}$ cannot be a constant. If g is a rational function, it is easy to see that $(g^n)^{(k)}(\zeta) - 1$ has zeros; if g is transcendental, then by Lemma 2, $(g^n)^{(k)}(\zeta) - 1$ also has zeros. Thus, there exists a ζ_0, such that

$$(g^n)^{(k)}(\zeta_0) = 1. \qquad \text{... (14)}$$

Then for large m and $\zeta \in \left\{ |\zeta - \zeta_0| = \frac{1}{m} \right\}$, there exists a positive constant $\varepsilon_0 > 0$, such that

$$|(g^n)^{(k)} - 1| \geq \varepsilon_0.$$

On the other hand, there exists a natural number $N > 0$ such that, for $j \geq N$,

$$|(f_j^n)^{(k)}(z_j + \rho_j \zeta) - (g^n)^{(k)}(\zeta)| < \varepsilon_0.$$

Thus, by Rouché's theorem, there exists a point $\zeta_j \in \left\{ |\zeta - \zeta_0| \le \frac{1}{m} \right\}$ such that

$$(f_j^n)^{(k)}(z_j + \rho_j \zeta_j) = 1.$$

Combining this with the hypothesis we have

$$\rho_j^{-\frac{k}{n}} |f_j(z_j + \rho_j \zeta_j)| \ge M \rho_j^{-\frac{k}{n}}. \qquad \text{... (15)}$$

Without loss of generality, we suppose that $\zeta_j \to \zeta_0'$, then $|\zeta_0' - \zeta_0| \le \frac{2}{m}$. Now the left side of (15) converges to $g(\zeta_0')$ and the right side of (15) tends to ∞ by the fact that $\rho_j \to 0$, we have $g(\zeta_0') = \infty$. Let $m \to \infty$, then $\zeta_0' = \zeta_0$. Thus $g(\zeta_0) = \infty$, which contradicts (14). The proof of Theorem 3 is complete.

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