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MONOTONE ITERATIVE METHOD FOR DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

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In this paper, we employ the method of upper and lower solutions coupled with the monotone iterative technique to obtain results of existence and approximation of solutions for periodic boundary value problems of differential equations with piecewise constant arguments.

Key Words : **Differential Equation with Piecewise Constant Arguments; Monotone Iterative Method; Upper and Lower Solutions**

1. INTRODUCTION

Differential equations with piecewise constant arguments (EPCA for short) are originated in [1, 5]. They are closely related to impulse and loaded equations and, especially, to difference equations of a discrete arguments. These equations have the structure of continuous dynamical systems within intervals of certain length. Continuity of a solution at a point joining any two consecutive intervals then implies recursion relations for the solution at such points. Many oscillatory properties of EPCA were mentioned, for example, see [1, 4, 5] and the references cited therein. In this paper we discuss the periodic boundary value problem (PBVP for short) of EPCA

$$x'(t) = f(t, x(t), x([t-1])), \quad t \in J, \qquad \dots (1)$$

$$x(0) = x(T), \qquad \dots (2)$$

where $J = [0, T]$ and $[\cdot]$ designates the greatest integer function. Let Ω denote the class of all functions $x : J \cup \{-1\} \to R$ satisfying that

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(*i*) $x(-1) = x(0)$;

(*ii*) $x(t)$ is continuous fot $t \in J$; and

(*iii*) $x'(t)$ exists on J with the possible exception at the integer points and $t = T$, where one side derivatives exist.

A function $\alpha \in \Omega$ is said to be a lower solution of (1) and (2), if it satisfies

$$\alpha'(t) \leq f(t, \alpha(t), \alpha([t-1])), \quad t \in J,$$

$$\alpha(0) \leq \alpha(T).$$

An upper solution for (1) and (2) is defined analogously by reversing the above inequalities.

We employ the method of upper and lower solutions coupled with the monotone iterative technique (see [2, 3] and the references therein) to establish the results of existence and approximation of solutions for PBVP of EPCA.

2. A Comparison Result

For the successful employment of the monotone iterative technique we need a certain comparison theorem. In this section, a general comparison theorem is developed.

Theorem 1 — *Suppose that* $m \in \Omega$ *such that*

$$m'(t) \leq -M_1 m(t) - M_2 m([t-1]), \quad t \in J, \qquad \text{... (3)}$$

$$m(0) \leq m(T), \qquad \text{... (4)}$$

where $M_1 > 0, M_2 \geq 0$ *are constants such that*

$$1 - \frac{M_2}{M_1} M e^{M_1} (e^{M_1} - 1) \geq 0, \qquad \text{... (5)}$$

where

$$M = \begin{cases} [T] + 1, & T \neq [T], \\ T, & T = [T]. \end{cases}$$

Then $m(t) \leq 0$ *for all* $t \in J$.

Proof : Set $p(t) = m(t)\, e^{M_1 t}$. Then the inequality (3) reduces to

$$p'(t) \leq -M_2 p([t-1])\, e^{M_1(t-[t-1])}.$$

Hence,

$$p(t) \leq p(n-1) - \frac{M_2}{M_1} p(n-2)\, (e^{M_1(t-n+2)} - e^{M_1})$$

for $t \in [n-1, n), n = 1, 2, \ldots, M-1,$ and

$$p(t) \le p(M-1) - \frac{M_2}{M_1} p(M-2)\,(e^{M_1(t-M+2)} - e^{M_1}), \quad \text{for } t \in [M-1, T].$$

We have the following two possibilities:

(a) $p(0) \le 0$; (b) $p(0) > 0$.

In case (a), we can have

$$p(n) \le (1 - nN)\,p(0), \quad n = 0, 1, 2, \ldots, M,$$

where $$N = \frac{M_2}{M_1} e^{M_1} (e^{M_1} - 1) \text{ and } p(n) \le 0.$$

In fact, when $n = 1$, one have

$$p(1) \le p(0) - p(-1)\,N = (1 - N)p(0)$$

and $p(1) \le 0$. Suppose that

$$p(i) \le (1 - iN)\,p(0), \quad i = 1, 2, .., k,$$

where $k < M$ and $p(i) \le 0$, then

$$\begin{aligned} p(k+1) &\le p(k) - N_p(k-1) \\ &\le (1-N)\,p(k-1) - N_p(k-2) \\ &\le (1-2N)\,p(k-2) - N_p(k-3) \\ &\quad \ldots\ldots\ldots\ldots \\ &\le (1-(k-1)N)p(1) - N_p(0) \\ &\le (1-(k-1)N)\,(1-N)p(0) - N_p(0) \\ &= (1-kN)p(0) - N_p(0) + (k-1)N^2\,p(0) \\ &\le (1-(k+1)N)p(0). \end{aligned}$$

By the inductive method one see

$$p(n) \le (1 - nN)\,p(0), \; = 1, 2, \ldots, M,$$

$$p(T) \le p(M-1) - N_p\,(M-2) \le (1 - MN)\,p(0)$$

and $p(n) \le 0$. For any $t \in J$, there exists an integer $n \le M-1$ such that $t \in [n-1, n)$ or $t \in [M-1, T]$. Hence

$$p(t) \le p(n-1) - \frac{M_2}{M_1} p(n-2)\,(e^{M_1(t+1-n)} - e^{M_1})$$

$$\leq p(n-1) - N_p(n-2) \leq 0.$$

That is $m(t) \leq 0$ for $t \in J$.

In the case (b), we need to consider two situations :

(i) There exists a positive integer k, $k < M$ such that $p(k) < 0$, $p(j) > 0$, $j = 0, 1, \ldots, k-1$.

(ii) $p(j) > 0$, $j = 1, 2, \ldots, M$.

If (i) holds, one can get that $p(i) \leq 0$ and

$$p(i+1) \leq (1 - iN)\, p(k) - N_p\,(k-1),\ i = k+1, \ldots, M-1.$$

Hence one have $p(T) < 0$. This leads to a contradiction to $m(0) \leq m(T)$.

If (ii) holds, then one get $p(t) > 0$, it is easy to see that $p(0) > p(T)$. That is a contradiction. The proof of Theorem 1 is complete.

If $m \in \Omega$ is a solution of PBVP

$$m'(t) + M_1 m(t) + M_2 m\,([t-1]) = 0,\ t \in J, \qquad \ldots (6)$$

$$m(0) = m(T), \qquad \ldots (7)$$

then both $m(t)$ and $-\,m(t)$ satisfy the inequality (3) and (4). Hence we have

Corollary 1 — Assume that (5) holds. Then the PBVP (6) and (7) has the unique solution $m(t) = 0$.

From the proof of Theorem 1, we can obtain

Crorollary 2 — Assume that (5) holds and $m \in \Omega$ satisfying (3) and

$$m(0) \leq 0.$$

Then $m(t) \leq 0$ for all $t \in J$.

3. Linear PBVPs

In this section, in order to develop the monotone iterative technique for (1) and (2), we consider the linear PBVP

$$x'(t) + M_1 x(t) + M_2 x([t-1]) = \sigma(t),\ t \in J, \qquad \ldots (8)$$

and $$x(0) = x(T), \qquad \ldots (9)$$

where M_1, M_2 are constants and $\sigma(t)$ is piecewise continuous and bounded in J.

Theorem 2 — *Assume that $M_1 > 0, M_2 \geq 0$ such that (5) and*

$$1 - M_2\, e^{M_1 T} > 0 \qquad \ldots (10)$$

hold. Then PBVP (8) and (9) has a unique solution.

PROOF : Let

$$w_0 = \max_{t \in J} |\sigma(t)|, \; w_1 = \frac{w_0 e^{M_1 T}}{1 - M_2 e^{M_1 T}}$$

and define an operator $S : \Omega_1 \to \Omega$ as

$$(Sx)(t) = \frac{e^{-M_1 t}}{e^{M_1 T} - 1} \int_0^T (\sigma(s) - M_2 x([s-1])) e^{M_1 s} ds$$

$$+ e^{-M_1 t} \int_0^t (\sigma(s) - M_2 x([s-1])) e^{M_1 s} ds, \quad t \in J,$$

and $\quad (Sx)(-1) = (Sx)(0),$

where $\quad \Omega_1 = \{x \in \Omega : |x| \le w_1, x(0) = x(T)\}.$

It is easy to see that Ω_1 is a closed bounded convex set and $S : \Omega_1 \to \Omega_1$ is compact according to Ascoli-Arzela's theorem. Hence, there exists a solution of PBVP (8) and (9) by Schauder's fixed point theorem. The uniqueness of solutions of PBVP (8) and (9) follow from Theorem 1. In fact, suppose that x and y are two distinct solutions of (8) and (9) and let $m(t) = x(t) - y(t)$, then $m(t)$ satisfies (6) and (7). Hence, by Corollary 1 we have $m(t) = 0$. The proof of Theorem 2 is complete.

A function $v \in \Omega$ is said to be a lower solution for (8) and (9) if it satisfies

$$v'(t) + M_1 v(t) + M_2 v([t-1]) \le \sigma(t) \; t \in J \qquad \text{... (11)}$$

and

$$v(0) \le v(T). \qquad \text{... (12)}$$

An upper solution for (8) and (9) is defined analogously by reversing the inequalities of (11) and (12).

***Theorem* 3** — *Let v and w be lower and upper solutions of (8) and (9) such that $v \le w$ on J and assume that (5) is satisfied. Then (8) and (9) has a unique solution $x \in [v, w]$.*

PROOF : For each $a \in [v(0), w(0)]$, denote by $x(\cdot; a)$ the unique solution of (8) with $x(-1) = x(0) = a$. Using Corollary 2, it is easy to see that $v(t) \le x(t; a) \le w(t)$ for $t \in J$ and $x(t; a_1) \le x(t; a_2)$ on J for $a_1, a_2 \in [v(0), w(0)]$ with $a_1 \le a_2$. Hence

$$v(0) \le v(T) \le x(T; v(0)) \le x(T; w(0)) \le w(T) \le w(0).$$

Denote sequence $\{v_n(t)\}$ by

$$\begin{cases} v_n(t) = x(t; v_{n-1}(T)), n = 1, 2, \ldots, \\ v_0(t) = v(t), \end{cases} \quad t \in J.$$

By induction we can see that the sequence satisfies

$$v_0(t) \le V_1(t) \le V_2(t) \le \ldots \le V_n(t) \ldots \le V_w(t),$$

and is convergent. Let $y(t) = \lim_{n \to \infty} v_n(t)$. Notice that

$$\begin{cases} v_n(t) = v_n(0) + \int_0^t (\sigma(s) - M_2 v_n([s-1]))\, e^{-M_1(t-s)}\, ds, \ t \in J \\ v_n(0) = v_{n-1}(T), \end{cases}$$

by Lebesgue Dominate Convergence Theorem, we know that $y(t)$ is the solution of (8) and (9).

The uniqueness of the solution of (8) and (9) can be obtained from Corollary 1. The proof of Theorem 3 is complete.

4. Monotone Method

We are now in a position to prove the results concerning the external solutions of (1) and (2).

Theorem 4 — *Let α and β be lower and upper solutions of (1) and (2) such that $\alpha \le \beta$ on J. Assume that $f \in C[J \times R^2]$ such that*

$$f(t, v_1, w_1) - f(t, v_2, w_2) \ge -M_1(v_1 - v_2) - M_2(w_1 - w_2) \quad \ldots (13)$$

for $t \in J$ and $v_i, w_i \in R\ (i = 1, 2)$ with $\alpha(t) \le v_2 \le v_1 \le \beta(t)$, $\alpha(t-1]) \le w_2 \le w_1 \le \beta([t-1])$, where M_1 and M_2 satisfy (5). Then, there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that $\alpha_n \to \rho$, $\beta_n \to r$ as $n \to \infty$ uniformly and monotonically on J and that $\rho, r \in \Omega$ are minimal and maximal solutions of PBVP (1) and (2) respectively.

Proof : For any $y \in \Omega$ with $\alpha \le y \le \beta$, let

$$\sigma(t) = f(t, y(t), y([t-1])) + M_1\, y(t) + M_2 y([t-1]),$$

then we have that

$$\alpha'(t) + M_1\, \alpha(t) + M_2\, \alpha([t-1])$$

$$\le f(t, \alpha(t), \alpha([t-1])) + M_1 \alpha(t) + M_2 \alpha[[t-1])$$

$$\le f(t, y(t), y([t-1])) + M_y\, y(t) + M_2\, y([t-1]) = \sigma(t)$$

and $\beta'(t) + M_1\, \beta(t) + M_2\, \beta([t-1]) \ge \sigma(t)$. As a consequence, α and β are respectively a lower and an upper solution for (8) and (9). Thus PBVP (8) and (9) has a unique solution $x \in [\alpha, \beta]$ by Theorem 3.

Now define a mapping A by $Ay = x$ where for any $y \in \Omega$ with $\alpha \le y \le \beta$, x is the unique solution of (8) and (9). First we shall that $\alpha \le A\alpha$ and $A\beta \le \beta$. Set $m(t) = \alpha(t) - A\alpha(t)$. From (8), (9) and the definition of lower solution we know that $m(t)$ satisfies (3) and (4). Hence, $m(t) \le 0$ due to Theorem 1. Similarly we can prove that $A\beta \le \beta$. Next we shall show that $Ay_1 \le Ay_2$ for any y_1 and $y_2 \in \Omega$ with $\alpha \le y_1 \le y_2 \le \beta$. Set $m(t) = Ay_1(t) - Ay_2(t)$. Using (8), (9) and (13), we obtain

$$m'(t) \leq -M_1 m(t) - M_2 m[t-1]),\ m(0) = m(T)$$

which implies by Theorem 1, that $Ay_1 \leq Ay_2$. It is therefore easy to see that the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$ can be defined by

$$\alpha_{n+1} = A\alpha_n \text{ and } \beta_{n+1} = A\beta_n$$

and the iterates satisfy

$$\alpha \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_1 \leq \beta$$

on J. It follows, from standard arguments (see [2]), that $\lim_{n \to \infty} \alpha_n(t) = \rho(t)$ and $\lim_{n \to \infty} \beta_n(t) = r(t)$ uniformly on J and $\rho(t)$ and $r(t)$ are solutions of (1) and (2).

Finally, to prove that ρ is the minimal solution on $[\alpha, \beta]$, let x be any solution of (1) and (2) on $[\alpha, \beta]$. It is obvious that $\alpha_0 \leq x$. Now if $\alpha_n \leq x$, one can easily see that $\alpha_{n+1} \leq x$ by considering the function $\phi = x - \alpha_{n+1}$ and applying Theorem 1 again. Thus, passing to the limit, we may conclude that $\rho \leq x$.

The same arguments prove that $x \leq r$. The proof of the Theorem 4 is complete.

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