Indian J. pure appl. Math., **31**(1) : 77-84, January 2000

STATISTICAL h-REGULAR SERIES METHODS IN TOPOLOGICAL GROUPS

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(*Received* 24 *December* 1998; *Accepted* 1 *April* 1999)

In this paper, the concept of statistical h-regularity of series methods is introduced in topological groups, and it is proved that a series method is statistically h-regular if and only if it is statistically generalized h-multiplicative.

Key Words : Statistical Convergence; Series; Summability; Topological Groups

1. INTRODUCTION

In [17], Prullage first introduced the concept of summability in topological groups and he continued in his papers, [18], [19] and [20], and the concept was investigated by Çakalli in the papers [2], [3], [4], [5], [6], [7], [8] and [9], and Çakalli and Thorpe in [10] and [11].

In [12], A complex number sequence $x=(x_n)$ is called statistically convergent to the number L if for each $\varepsilon>0$

$$\lim_{m\to\infty} m^{-1} \mid \{n\le m : \mid x_n - L \mid \ge \varepsilon\} \mid = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. This concept was also studied by Fridy[13] and Maddox[15&16], was generalized to Banach spaces by Kolk[14] and to locally convex spaces by Maddox[16]. Recently, Çakalli generalized the concept to topological groups and investigated statistical summability in topological groups in his paper[8]. Quite recently, Çakalli investigated statistical generalized multiplicative series methods in topological groups in [9].

The purpose of this paper is to introduce statistical h-regularity of series methods in topological groups and prove that a triangular series-to-sequence method in a metrizable abelian topological group is statistically h-regular if and only if it is statistically generalized h-multiplicative.

2. DEFINITIONS AND NOTATION

Throughout this paper, X will denote an abelian topological Hausdorff group written additively, which satisfies the first axiom of countability. For a subset A of X, $s(A)$ will denote the class of all sequences $(x(n))$ such that $x(n) \in A$ for $n = 1, 2, \ldots$; $c(X)$, $c_0(X)$, $C(X)$, $\gamma(X)$ and $\Gamma(X)$ will denote the set of all convergent sequences, the set of all convergent-to-zero sequences, the set of all Cauchy sequences, the set of all convergent series and the set of all Cauchy series in X, respectively.

In [8], a sequence $(x(k))$ is called statically convergent to an element l of X if for each neighbourhood U of 0,

$$\lim_{m\to\infty} m^{-1} |\{n \le m : x(k) - l \in U\}| = 0.$$

In case a sequence $(x(n))$ is statistically convergent to l we write S-lim $x(n) = l$. The set of all statistically convergent sequences in X is denoted by $S(X)$, and the set of all statistically convergent-to-zero sequences in X is denoted by $S_0(X)$. To avoid the confusion recall that $s(X)$ is the set of all sequences in X. Naturally, we define statistical convergence of a series $\Sigma x(n)$ in topological groups as follows :

We say that a series $\sum_{n=1}^{\infty} x(n)$ in X statistically convergent to an element l of X if for each neighbourhood U of 0,

$$\lim_{m\to\infty} m^{-1} \left| \left\{ n \le m : \left(\sum_{k=1}^{n} x(k) \right) - l \notin U \right\} \right| = 0.$$

In this case, we write $S - \Sigma x(n) = l$ or $\Sigma x(k) = l(S)$. By $\gamma^S(X)$, we denote the set of all statistically convergent series in X. In particular, by $\gamma_0^S(X)$, we will denote the set of all statistically convergent-to-zero series in X ([9]).

It is clear that $\gamma(X) \subset \gamma^S(X)$, and the inclusion is strict. For example, for an element x different from zero, the series $\Sigma x(n)$ defined by

$$x(n) = \begin{cases} (-1)^{m-1} x & \text{if } n = m^2 \text{ for some } m = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

is statistically convergent to zero but not convergent at all.

A sequence of additive functions $f(m)$ whose domain, for each m, is some subset of $s(X)$ and contains the set $\Gamma(X)$, and whose range is contained in X is said to be a series method. The sequence $(f(m : x(1), x(2), \dots, x(m), \dots))$ is called the transformed sequence of the series $\Sigma x(n)$ by f, and the sequence $f = (f(m))$ is called a series-to-sequence method on X. The series

$$\Sigma g(k : x(1), x(2), \dots, x(k), \dots)$$

is said to be the transformed series of the series, $\Sigma x(n)$, by the series method $g = (g(k))$, and the sequence $g = (g(k))$ is called a series-to-series method on X. The intersection of the domains of the functions is called the domain of the series method. If the domain of $(f(m))$ is all of $s(X)$, and the equality

$$f(m : x(1), x(2), \dots, x(m), \dots) = f(m : y(1), y(2), \dots, y(m), \dots)$$

holds for every positive integer m and every pair of sequences $(x(n))$, $(y(n))$ for which $x(n) = y(n)$ for $n = 1, 2, ..., m$, then the series method $(f(m))$ is said to be triangular. In case $(f(m))$ is a triangular series method, we will write $f(m : x(1), x(2), ..., x(m))$ instead of $f(m : x(1), x(2), ..., x(m), ...)$. For the sake of simplicity, we write f instead of $(f(m))$ when no confusion arises[5].

We call a series-to-sequence method on X, $(f(m))$, is statistically generalized h-multiplicative if

$$S - \lim f(m : x(1), x(2), ..., x(m), ...) = h\left(\sum_{n=1}^{\infty} x(n)\right)$$

and a series-to-series method on X, $(g(m))$, is statistically generalized h-multiplicative if

$$S - \Sigma\, g(m : x(1), x(2), ..., x(m), ...) = h\left(\sum_{n=1}^{\infty} x(n)\right)$$

for each $(x(n)) \in \gamma(X)$ where h is a function from X to X[9].

Now we give the following definition :

We say that a series-to-sequence method on X, $(f(m))$, is statistically h-regular if for each neighbourhood U of 0, there exists a constant positive integer M such that

$$\lim_{n \to \infty} n^{-1} \left| \left\{ k \le n : h\left(\sum_{i=1}^{M} x(i)\right) - f(k : x(1), x(2), ..., x(k), ...) \notin U \right\} \right| = 0$$

for each $(x(n)) \in \Gamma(X)$, and we call a series-to-series method on X, $(g(m))$, is statistically h-regular if for each neighbourhood U of 0, there exists a constant positive integer M such that

$$\lim_{m \to \infty} m^{-1} \left| \left\{ k \le m : h\left(\sum_{i=1}^{M} x(i)\right) - \sum_{i=1}^{k} g(i : x(1), x(2), ..., x(i), ...) \notin U \right\} \right| = 0$$

where h is a continuous function from X to X.

2. Summability

Here we give two lemmas analogous to one in [11] and prove a theorem giving necessary and sufficient conditions for a triangular series-to-sequence method to be h-regular. It turns out that a triangular series method is h-regular if and only if it is generalized h-multiplicative.

Lemma 1 — Every topological Hausdorff group, which satisfies the first axiom of countability, has a base of symmetric neighbourhoods of the origin (see page 24 in [1]).

Now we prove the following lemma :

Lemma 2 — A series-to-sequence method on X, $(f(m))$, is statistically h-regular if and only if it satisfies the following conditions :

(1) $S - \lim_{m \to \infty} f(m : 0, 0, ..., 0, x, 0, 0, ...) = h(x)$ for all x in X in any fixed position.

(C2) For each neighbourhood U of 0 and for each positive number ε, there is a neighbourhood V of 0 such that for all $(x(n)) \in C(X) \cap s(V)$, there exists a positive integer M such that $m > M$ implies that

$$m^{-1} \mid \{ k \leq m : f(k : \Delta x(1), \Delta x(2), ..., \Delta x(k), ...) - h(x(1)) \in U \} \mid < \varepsilon.$$

where $\Delta x(i) = x(i) - x(i+1)$ for each positive integer i.

PROOF : *Sufficiency* : Let $(x(k))$ be any element of $\Gamma(X)$ and write $s(n) = \sum_{i=1}^{n} x(i)$ for each positive integer n. Let W be any neighbourhood of 0 and ε a positive number. We may choose a symmetric neighbourhood U of 0 such that $U + U + U \subseteq W$. By condition (C2), there exists a neighbourhood V of 0 such that for all Cauchy sequences in V, there is a positive integer N so that $m > N$ implies that

$$m^{-1} \mid \{ k \leq m : f(k : \Delta x(1), \Delta x(2), ..., \Delta x(k), ...) - h(x(1)) \notin U \} \mid < \varepsilon/2.$$

Since $(s(n))$ is a Cauchy sequence, there is a positive integer M such that $m, n \geq M$ implies that $x(n) - x(m) \in V$. Then the sequence

$$(0, 0, ..., 0, s(M+1) - s(M), s(M+2) - s(M), s(M+3) - s(M), ...)$$

is in $C(X) \cap s(V)$, where 0 appears $M + 1$ times, so there exists an integer $M(1) > M$ such that $m > M(1)$ implies that

$$m^{-1} \mid \{ k \leq m : f(k : 0, 0, ..., 0, -x(M+1), -x(M+2), -x(M+3), ...) \notin U \} \mid < \varepsilon/2.$$

By condition (1), there exists an integer $M(2) > M(1)$ such that if $m > M(2)$, then

$$m^{-1} \mid \{ k \leq m : h(s(M)) - f(k : x(1), x(2), ..., x(M), 0, 0, ...) \notin U \} \mid < \varepsilon/2.$$

Thus if $m > M(2)$ then

$$m^{-1} \mid \{ k \leq m : h(s(M)) - f(k : x(1), x(2), ..., x(k), ...) \notin W \} \mid$$

$$\leq m^{-1} \mid \{ k \leq m : f(k: 0, 0, ..., 0, -x(M+1), -x(M+2), ...) \notin U \} \mid$$

$$+ m^{-1} \mid \{ k \leq m : h(s(M)) - f(k : x(1), x(2), ..., x(M), 0, 0, ...) \notin U \} \mid$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This completes the proof of the sufficiency.

Necessity — The necessity of (1) is obvious. To prove the necessity of (C2), let us take any neighbourhood U of 0. Choose a symmetric neighbourhood W of 0 such that $W + W \subseteq U$. Since h is continuous there exists a neighbourhood V of 0 such that $h(V) \subset W$. Let $(x(n))$ be in $C(X) \cap s(V)$. Take any positive number ε. Now consider the series $\Sigma \Delta x(n)$ which is a Cauchy series. By the assumption that f is statistically h-regular, where is a positive integer M such that $m > M$ implies that

$$m^{-1} \mid \{ k \leq m : f(k : \Delta x(1), \Delta x(2), ..., \Delta x(m), ...) - h(x(1)) - h(x(M)) \notin W \} \mid < \varepsilon.$$

Therefore, $m > M$ implies that

$$m^{-1} \mid \{ k \leq m : f(k : \Delta x(1), \Delta x(2), ..., \Delta x(k), ...) - h(x(1)) \notin U \} \mid < \varepsilon.$$

This completes the proof of the necessity, hence the proof of the lemma.

Lemma 3 — let $f = (f(m))$ be a series-to-sequence method on X satisfying the condition (1) of Lemma 2. Then the condition (C2) of Lemma 2 is equivalent to the following condition :

(2′) For each neighbourhood U of 0 and for each positive number ε, there is a neighbourhood V of 0 such that for all $(x(n)) \in C(X) \cap s(V)$, there exists a positive integer M such that $m > M$ implies that

$$m^{-1} \mid \{ k \leq m : f(k : \delta x(1), \delta x(2), ..., \delta x(m), ...) \notin U \} \mid < \varepsilon,$$

where $\delta x(n) = x(n) - x(n-1)$ for each positive integer n and $x(0) = 0$ for convention.

The proof of this lemma is easy so is omitted.

***Theorem* 4** — *A triangular series-to-sequence method on X, (f(m)), is statistically h-regular if and only if the following conditions are satisfied :*

(T1) $S\text{-}\lim_{m \to \infty} f(m: 0, 0, ..., 0, x, 0, 0, ..., 0) = h(x)$ *for all x in X in any fixed position.*

(T2) For each neighbourhood U of 0 and for each positive number ε, there exist a neighbourhood V of 0 and a positive integer N such that if m and k are integers satisfying $m > k > N$ and if $x(n)$ is in V for $n = k, k+1, ..., m$ then

$$m^{-1} \mid \{ i \leq m : f(i : 0, 0, ..., 0, \Delta x(k), \Delta x(k+1), \Delta x(k+2), ..., \Delta x(i)) \notin U \} \mid < \varepsilon.$$

PROOF : (*Sufficiency*) : Take any Cauchy series, $\Sigma x(k)$, write $s(n) = \sum_{k=1}^{n} x(k)$. Let W be any neighbourhood of 0. Choose a symmetric neighbourhood U of 0 such that $U + U \subseteq W$. Take any positive number ε. By condition (*T2*), there exist a neighbourhood V of 0 and a positive integer N such that if m and k are integers satisfying $m > k > N$ and if $x(n)$ is in V for $n = k, k+1, k+2, ..., m$ then

$$m^{-1} \mid \{ i \leq m : f(i : 0, 0, ..., 0, \Delta x(k), \Delta x(k+1), ..., \Delta x(i)) \notin U \} \mid < \varepsilon/2.$$

Since $(x(n)) \in \Gamma(X)$, there exists a positive integer $M > N$ such that $n, k \geq M$ implies that $s(n) - s(k) \in V$. Then the sequence

$$(0, 0, ..., 0, s(M+1) - s(M), s(M+2) - s(M), s(M+3) - s(M), ...)$$

is in $s(V)$. Therefore,

$$m^{-1}\,|\{i \le m : f(i : 0, 0, ..., 0, x(M+1), x(M+2), ..., x(i)) \notin U\}| < \varepsilon/2 \qquad ...\ (a)$$

From the condition (1), there exists a positive integer $M(1) > M$ such that for $m > M(1)$

$$m^{-1}\,|\{i \le m : f(i : x(1), ..., x(M), 0, 0, ..., 0) - h(s(M)) \notin U\}| < \varepsilon/2 \qquad ...\ (b)$$

It follows from (a) and (b) that $m > M(1)$ implies that

$$m^{-1}\,|\{i \le m : f(i : x(1), x(2), ..., x(i)) - h(s(M)) \notin W\}| < \varepsilon$$

Hence, it follows that f is statistically h-regular.

Necessity — The necessity of $(T1)$ is obvious. We are going to prove that $(T2)$ is also necessary. Suppose that $(T2)$ is not satisfied so that there exist a neighbouhood U of 0 and a positive number ε such that for all neighbourhoods V of 0 and all positive integers N, there are integers m and k satisfying the condition $m > k > N$ and there are elements $y(n)$ in V for $n = k, k + 1, ..., m$ such that

$$m^{-1}\,\left|\left\{i : k \le i \le m \text{ and } f(i : 0, 0, ..., 0, \Delta y(k), \Delta y(k+1), ..., \Delta y(i)) \notin U\right\}\right| \ge \varepsilon.$$

Choose a symmetric neighbourhood W of 0 such that $W + W \subseteq U$. Take any neighbourhood V of 0. Let $(V(1))$ be a base of neighbourhoods of 0 such that $V(1) \subseteq V \cap W$, $V(i+1) \subseteq V(i)$ for $i = 1, 2, ...$ and

$$\bigcap_{i=1}^{\infty} V(i) = \{0\}.$$

Choose an integer $m(1)$ and elements $x(n)$ in V for $n = 1, 2, ..., m(1)$ arbitrarily. Now suppose that an increasing sequence of integers $m(i)$ $(i = 1, 2, ..., r)$ and elements $x(n)$ for $n = 1, 2, ..., m(r)$ have been constructed. By condition $(T1)$, there is an integer $n(r + 1) > m(r)$ so that if $m > n(r + 1)$ then

$$m^{-1}\,\left|\left\{k \le m : f(k : \Delta x(1), ..., \Delta x(m(r)), 0, ..., 0) - h(x(1)) \notin W\right\}\right| < \varepsilon/2.$$

Also there exist integers $m(r + 1)$ and $k(r + 1)$ such that $m(r + 1) > k(r + 1) > n(r + 1)$ and there exist elements $x(n)$ in $V(r)$ $(n = k(r + 1), k(r + 1) + 1, ..., m(r + 1))$ so that

$$(m(r+1))^{-1}\,|\,\{k \le m(r+1) : f(k : 0, ..., 0, \Delta x(k(r+1)),$$

$$\Delta x(k(r+1)+1) ..., \Delta x(k)) \notin U\}\,| \ge \varepsilon.$$

Finally choose $x(n) = 0$ for $n = m(r) + 1, m(r) + 2, ..., k(r + 1) - 1$. Since $(V(i))$ is a nested base and $x(n)$ is in $V(r)$ for $n = k(r + 1), k(r + 1) + 1, ...,$ it follows that $(x(n)) \in c_0(X)$. Choose an integer r such that $m(r + 1) > N$. Then $m(r + 1) > N$ implies that

$$(m(r+1))^{-1}\,\left|\left\{i \le m(r+1) : f(i : \Delta x(1), \Delta x(2) ..., \Delta x(i)) - h(x(1)) \notin W\right\}\right|$$

$$\geq (m(r+1))^{-1} \left| \{ i \leq m(r+1) : f(i : 0, ..., 0, \Delta x(k(r+1)), \Delta x(k(r+1)+1) ..., \Delta x(i)) \notin U \} \right|$$

$$- (m(r+1))^{-1} \left| \{ i \leq m(r+1) : f(i : \Delta x(1), ..., \Delta x(m(r)), 0, 0, ..., 0) - h(x+1)) \notin U \} \right|$$

$$> \varepsilon - \varepsilon/2 = \varepsilon/2.$$

Thus condition (2) is not satisfied. This contradiction completes the proof of the neçessity and hence the proof of the theorem.

The following theorem was given in [9] :

***Theorem* 5** — *A triangular series-to-sequence method on X, (f(m)), is statically generalized h-multiplicative if and only if it satisfies the conditions (T1) and (T2).*

Thus we have the following :

Corollary 6 — A triangular series-to-sequence method on X, f, is statistically h-regular if and only if it is statistically generalized h-multiplicative.

Now we will give a theorem for series-to-series methods. First we give the following relations between a series-to-sequence method $(f(m))$ and a series-to-series method $(g(m))$:

Let $g = (g(k))$ be a series-to-series method on X. Define the associated series-to-sequence method on X, $f = (f(k))$, by the equations :

(*I*) $g(1 : x) + g(2 : x) + ... + g(k : x) = f(k : x)$ for each sequence $x = (x(n))$ in the domain of $(g(i))$ for $i = 1, 2, ..., k$,

(*II*) $g(1 : x) = f(1 : x)$, $g(k : x) = f(k : x) - f(k - 1 : x)$ for $k = 1, 2, ...$ where $g(i : x)$ and $f(i : x)$ are written instead of $g(i : x(1), x(2), ...)$ and $f(i : x(1), x(2), ...)$ for $i = 1, 2, ...,$ respectively.

***Theorem* 7** — *If f and g are connected by the equations (I) and (II), then the series-to-series method g is statistically h-regular if and only if the series-to-sequence method f is statistically h-regular.*

The proof of this theorem is obvious, therefore omitted.

Corollary 8 — A series-to-series method g is statistically h-regular if and only if it is statistically generalized h-multiplicative.

It should be noted that the results for the series methods which may not be necessarily triangular are also valid in non-metrizable topological groups.

It should also be noted that the study in this paper can be adopted to the case where the series methods are from an abelian first countable topological Hausdorff group to an abelian topological Hausdorff group.

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