Indian J. pure appl. Math., **31**(1) : 85-91, January 2000

LANCZOS SPINTENSOR AND LIÉNARD WIECHERT FIELD

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(*Received* 26 *November* 1998; *Accepted* 1 *April* 1999)

This paper proves that the analogies between the Lanczos spintensor K_{abc} for the conformal tensor of the spacetime and the Weert potential $\underset{B}{K}_{ijc}$ for the bounded part of the Liénard-Wiechert field allows to construct generators for the last one, which is important for the electrodynamics of classical charged particles.

Key Words : Lanczos Superpotential; Splitting and Generators for the Weert Potential; Potentials for the Liénard-Wiechert Field

1. INTRODUCTION

In all this paper, we shall employ the quantities and notation given in d{italic eta}il in López-Bonilla *et al*[1,2].

A point charge q in arbitrary motion in Minkowski space produces the electro-magnetic field of Liénard-Wiechert (LW)[3], with the 4-potential:

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$$A_r = qw^{-1} v_r \qquad \dots (1)$$

which generates the Faraday tensor via the expression[4]

$$F_{ij} = A_{j,i} - A_{i,j} = qw^{-2} U_i \times k_j. \qquad \dots (2)$$

Plebański[5] proved that the eq. (2) is equivalent to:

$$F_{ij} = q\tau_{,i} \times B_{,j}, \qquad \dots (3)$$

where it is easier to figure out the fulfillment of the Maxwell equations :

$$F^{j}_{i,j} = 0 \qquad \dots (4)$$

and

$$F_{ij,r} + F_{jr,i} + F_{ri,j} = 0. \qquad \dots (5)$$

Gaftoi *et al.*[6] determined an interesting null eigenvector η_j for F_{ac}

$$F^{c}_{i}\, \eta_c = -qw^{-2}\, \eta_i,\ \eta^c = U^c + \frac{1}{2}(a^2 - B^2)k^c,\ \eta^r\, \eta_r = 0, \qquad \dots (6)$$

with the corresponding Maxwell tensor given by[7]

$$T_{bc} = q^2 w^{-4}\left(\frac{1}{2} g_{bc} + \eta_b k_c + \eta_c k_b\right), \qquad \dots (7)$$

which is a more compact expression than existing ones[3,4,8].

Eq. (7) admits the Teitelboim splitting[1,2,9,10]

$$T_{ij} = \underset{B}{T_{ij}} + \underset{R}{T_{ij}} \qquad \dots (8)$$

such that

$$\underset{B}{T_{ij}} = q^2 w^{-4}\left[\frac{1}{2} g_{ij} + (k_i U_j + k_j U_i) - w^{-2}(1 - 2W)k_i k_j\right], \text{ Bounded part} \qquad \dots (9)$$

and

$$\underset{R}{T_{ij}} = q^2 w^{-4}(a^2 - w^{-2} W^2)k_i k_j, \text{ Radiative part} \qquad \dots (10)$$

which satisfies the conservation laws

$$\underset{B}{T^{j}_{c,j}} = 0 \qquad \dots (11)$$

and

$$\underset{R}{T^{j}_{c,j}} = 0, \qquad \dots (12)$$

off the world line. Weert[11] showed that eq. (11) implies the existence of the superpotential $\underset{B}{K}_{abc}$ playing the role of a generator for the bounded part:

$$\underset{B}{T}_{jc} = \underset{B}{K}_{j\ c,b}^{\ b} \qquad \ldots (13)$$

where

$$\underset{B}{K}_{jbc} = \frac{q^2}{4} w^{-4} [w^{-1} (4W - 3) (v_j \times k_b) k_c - 4(a_j \times k_b) k_c + g_{cb} k_j - g_{cj} k_b], \qquad \ldots (14)$$

which has[12] the same symmetrics that the Lanczos spintensor[13-19] for the conformal tensor of the spacetime:

$$\underset{B}{K}_{ijc} = -\underset{B}{K}_{jic}, \ \underset{B}{K}_{ijc} + \underset{B}{K}_{jci} + \underset{B}{K}_{cij} = 0, \qquad \ldots (15)$$

$$K_{i\ c}^{\ c} = 0 \qquad \text{Lanczos algebraic gauge} \qquad \ldots (16)$$

and

$$\underset{B}{K}_{ij,\ c}^{\ \ c} = 0, \qquad \text{Lanczos differential gauge.} \qquad \ldots (17)$$

With these symmetries López-Bonilla *et al*[12]. elucidated the physical meaning of the Weert potential, Aquino *et al.*[20] constructed a "Petrov Classification"[21] for the LW field, and López-Bonilla *et al.*[22] obtained the López splitting[23]:

$$\underset{B}{K}_{abc} = \underset{B}{\overline{K}}_{abc} + \underset{B}{\tilde{K}}_{abc}\ , \qquad \ldots (18)$$

which is very important to understand the angular momentum radiation of the point charge. Thus we see that eqs. (15, 16, 17) are useful ones into the analysis of the LW field.

On the other hand, López-Bonilla et al[24] reported a Lanczos spintensor for a rotating black hole[21,25,26], which has the remarkable structure:

$$K_{abc} = L_{ca;b} - L_{cb;a}, \qquad \ldots (19)$$

where the symmetric tensor L_{ij} appears as a more essential generator for the conformal tensor in Kerr geometry. Besides, Lanczos[13] obtained a tensor K_{ijr} following (19) for the case of weak gravtational fields, and also tried[27] to construct spintensors *via*

$$K_{abc} = Q_{ab}\, Q_c,\ Q_{ab} = -Q_{ba} \qquad \ldots (20)$$

in quadratic gravitational theories.

In this paper, $\underset{B}{\overline{K}}_{abc}$ has the structure given in (19), where L_{ij} is a radiation tensor in the sense of Villarroel[10, 28], and that $\underset{B}{\tilde{K}}_{abc}$ has the form (20) where Q_{ij} is the antisymmetrized product of two gradients similar to eq. (3).

2. Analysis of the Weert Potential

We tried to collect the terms of eq. (4) into two blocks such that everyone has the symmetries (15, 16, 17), thereby found[23] only one possibility for having (18) with

$$\underset{B}{\bar{K}}_{bjc} = \frac{q^2}{4} w^{-4} [3w^{-1} (v_j \times k_b) k_c + g_{cj} k_b - g_{cb} k_j], \qquad \dots (21)$$

$$\underset{B}{\tilde{K}}_{bjc} = q^2 w^{-4} [w^{-1} W(v_b \times k_j) - (a_b \times k_j)] k_c, \qquad \dots (22)$$

this implying a corresponding splitting in eq. (9) due to eq. (13):

$$\underset{B}{T}_{ij} = \underset{B}{\bar{T}}_{ij} + \underset{B}{\tilde{T}}_{ij} \quad , \qquad \dots (23)$$

where

$$\underset{B}{\bar{T}}_{bc} = \underset{B}{\bar{K}}_b{}^j{}_{c,j} = q^2 w^{-4} \left[\frac{1}{2} g_{bc} + w^{-1} (v_b k_c + v_c k_b) - w^{-2} (1 + 2W) k_b k_c \right], \qquad \dots (24)$$

and

$$\underset{B}{\tilde{T}}_{bc} = \underset{B}{\tilde{K}}_b{}^j{}_{c,j} = q^2 w^{-4} [(k_b a_c + k_c a_b) - w^{-1} W(v_b k_c + v_c k_b) + 4w^{-2} W k_b k_c] \qquad \dots (25)$$

and becomes evident that also eqs. (24, 25) satisfy the eq. (11) off the world line.

As an original result, the potential (21) has the same structure as (19), with

$$L_{bc} = \frac{q^2}{4} w^{-4} k_b k_c, \quad L^r_r = 0. \qquad \dots (26)$$

which is a "radiation tensor" because it verifies the Villarroel's conditions[28].

$$L_b{}^c{}_{,c} = 0, \quad L_{bc} k^c = 0. \qquad \dots (27)$$

It is interesting to note how a radiation tensor participates as generator of the bounded part of LW field. A similar situation appears in Gaftoi *et al*[10]: the Schott term[29] occurring in the Lorentz-Dirac equation[3,4,30-34] is originated from a radiation tensor, which is an uncommon fact because the usual procedure[9] is to associate the Schott term with the reversible emission of energy.

If we put the eqs. (19,27) into eq. (24) we find that $\underset{B}{\bar{T}}_{bc}$ is straightforward generated from L_{bc} via the D'Alembertian operator:

$$\underset{B}{\bar{T}}_{bc} = L_{bc}{}^{,r}{}_{,r} \equiv \Box L_{bc}. \qquad \dots (28)$$

Eq. (22) express that $\underset{B}{\tilde{K}}_{bjc}$ has the structure (20) in analogy with the Lanczos spintensor, with

$$Q_{bj} = qw^{-2}\,[w^{-1}W(v_b \times k_j) - (a_b \times k_j)],\quad Q_c = qw^{-2}\,k_c, \qquad \text{... (29)}$$

Q_r is a proper null vector of Q_{ij} because $Q_{ac}Q^c = 0$. It is easy to prove the relations:

$$Q_{ab,c} + Q_{bc,a} + Q_{ca,b} = 0,\ \det\,(Q_{ab}) = 0. \qquad \text{... (30)}$$

which are also verified by the Faraday tensor F_{ab} given in eq. (2). Now we remember the Stachel Theorem[35]:

"If $G_{ab} = -G_{ba}$ satisfies the eq. (30), then exist the functions ϕ and ψ such that

$$G_{ab} = \phi_{,a} \times \psi_{,b}\text{"}, \qquad \text{... (31)}$$

by this Theorem, F_{ab} has the form (3) with $\phi = q\tau$ and $\psi = B$. Therefore, Q_{ab} has the structure (31) with $\phi = q\tau$ and $\psi = w^{-1}\,(W-2)$, that is:

$$Q_{ij} = q\tau_{,i} \times [w^{-1}\,(W-2)]_{,j} \qquad \text{... (32)}$$

which is an original information on $\underset{B}{\tilde{K}}_{bjc}$

Besides, the eq. (32) allow to express Q_{ij} in the interesting form:

$$Q_{ij} = R_{j,i} - R_{i,j},\quad R_c = qw^{-1}p_c, \qquad \text{... (33)}$$

i.e. R_c is a "4-potential" for Q_{ab} in analogy with A_r and F_{rc}. The vector R_j was studied by Miglietta[36] in another context.

With respect to the radiative part[1,37,38] given by eq. (10: the eq. (12) leads to a non-local superpotential $\underset{R}{K}_{abc}$ generator of $\underset{R}{T}_{ab}$ *via* the expression

$$\underset{R}{T}_{jc} = \underset{R}{K}_{j\ c,b}^{\ b} \qquad \text{... (34)}$$

such that[39, 40]

$$\underset{R}{K}_{jbc} = F_{jb}G_c \qquad \text{... (35)}$$

with

$$G_c = qp_{(\theta)}\, p_{(\theta)} \left[\int_0^{\tau} a_{(\sigma)}\, a_{(\theta)} v_c d\tau + p_{(b\eta)} \int_0^{\tau} a_{(\sigma)} a_{(\theta)} e_{(b\eta)_c}\, d\tau \right] -$$

$$- q \left[\int_0^{\tau} a^2 v_c d\tau + p_{(\sigma)} \int_0^{\tau} a^2 e_{(\sigma)_c}\, d\tau \right], \qquad \text{... (36)}$$

thus $\underset{R}{K}_{abc}$ also satisfies eq. (20) as certain Lanczos spintensors.

We must indicate that the eqs. (9, 10) do not suffer changes if we add to $\underset{B}{K}_{abc}$ or/and $\underset{R}{K}_{abc}$ the potential:

$$K_{bjc} = q^2 w^{-3} \left[W(v_j \times k_b)\, v_c + (a_j \times k_b)\, (- w v_c + k_c) + W(g_{cb} k_j - g_{cj} k_b) \right] \qquad \text{... (37)}$$

which has the property $K^{j}_{bc\,j} \equiv 0$; however, with the use of eq. (37) the fulfillment of eqs. (15, 16, 17, 29, 20) is broken and we lose the analogies with the Lanczos superpotential.

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