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MODIFIED BESSEL-TYPE FUNCTION AND SOLUTION OF DIFFERENTIAL AND INTEGRAL EQUATIONS

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The paper is devoted to investigate the function

$$\lambda_{\gamma,\sigma}^{(\beta)}(z)=\frac{\beta}{\Gamma(\gamma+1-1/\beta)}\int_1^\infty (t^\beta-1)^{\gamma-1/\beta}t^\sigma e^{-zt}\,dt$$

with positive β and complex γ, σ and z such that $Re\ (\gamma)>(1/\beta)-1$ and $Re(z)>0$. When $\beta=2$ and $\sigma=0$, $\lambda_{\gamma,0}^{(2)}(z)=2^{\gamma+1}\pi^{-1/2}z^{-\gamma}K_{-\gamma}(z)$, where $K_{-\gamma}(z)$ is the modified Bessel function of the third kind. Special case is discussed when $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ is expressed in terms of the Tricomi confluent hypergeometric function $\Psi(a, c; x)$. The representation of $\lambda_{\gamma,\sigma}^{(\beta)}(x)$ via fractional integration operators are considered. Recurrence relations are proved. Applications are given to solve differential equations of fractional order and Abel-Volterra integral equations of the third kind.

Key Words : Bessel-type Function; Confluent Hypergeometric Function; Fractional Calculus Operators; Differential Equations of Fractional Order; Integral Equation

1. INTRODUCTION

The paper deals with the function

$$\lambda_{\gamma,\sigma}^{(\beta)}(z)=\frac{\beta}{\Gamma(\gamma+1-1/\beta)}\int_1^\infty (t^\beta-1)^{\gamma-1/\beta}t^\sigma e^{-zt}\,dt, \qquad \text{... (1.1)}$$

where

$$\beta>0;\ \gamma\in C, Re(\gamma)>\frac{1}{\beta}-1;\ \sigma\in \boldsymbol{R};\ z\in C, Re(z)>0, \qquad \text{... (1.2)}$$

C being the set of complex numbers. This function is analytic with respect to z. When $\sigma=0, \beta=1$ and $\sigma=0, \beta=2$, then

$$\lambda_{\gamma, 0}^{(1)}(z)=z^{-\gamma} e^{-z} \quad \ldots (1.3)$$

and

$$\lambda_{\gamma, 0}^{(2)}(z)=2^{\gamma+1} \pi^{-1/2} z^{-\gamma} K_{-\gamma}(z), \quad \ldots (1.4)$$

where $K_{-\gamma}(z)$ is the modified Bessel function of the third kind or McDonald function [4, Section 7.2.2].

The function $\lambda_{\gamma, \sigma}^{(\beta)}(z)$ was introduced by Glaeske, Kilbas and Saigo[5]. It is a modification of the function

$$\lambda_{\gamma}^{(n)}(z)=\frac{(2\pi)^{(n-1)/2}\sqrt{n}}{\Gamma(\gamma+1-1/n)}\left(\frac{z}{n}\right)^{\gamma n} \int_0^{\infty} (t^n-1)^{\gamma-1/n} e^{-zt}\, dt \quad \ldots (1.5)$$

$$\left(n \in N;\ \mathrm{Re}(\gamma)>\frac{1}{n}-1,\ Re(z)>0\right)$$

studied by Kratzel[7-9]. It should be noted that the function (1.5), being invariant with the accuracy of indice with respect to the usual differentiation[7-9], do not have such a property for the fractional differentiation. For this reason the function (1.1) is more preferable. Namely in [5] the relations

$$(I_{-}^{\alpha} \lambda_{\gamma, \sigma}^{(\beta)})(x)=\lambda_{\gamma, \sigma-\alpha}^{(\beta)}(x) \quad \ldots (1.6)$$

and

$$(D_{-}^{\alpha} \lambda_{\gamma, \sigma}^{(\beta)})(x)=\lambda_{\gamma, \sigma+\alpha}^{(\beta)}(x) \quad \ldots (1.7)$$

were proved. These formulae connect the functions (1.1) with different indices σ and $\sigma \pm \alpha$ by means of the Liouville right-sided fractional integration and differentiation operators I_{-}^{α} and D_{-}^{α} of order $\alpha>0$ defined by [13, section 5.1]

$$(I_{-}^{\alpha} \varphi)(x)=\frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{\varphi(t)dt}{(t-x)^{1-\alpha}} \, (x>0; \alpha>0) \quad \ldots (1.8)$$

and

$$(D_{-}^{\alpha} \varphi)(x)=\frac{1}{\Gamma(1-(\alpha y))}\left(-\frac{d}{dx}\right)^{[\alpha]+1} \int_x^{\infty} \frac{\varphi(t)\, dt}{(t-x)^{(\alpha y)}} \, (x>0; \alpha>0), \quad \ldots (1.9)$$

where $[\alpha]$ and $\{\alpha\}$ are integral and fractional parts of α. Asymptotic behavior of $\lambda_{\gamma, \sigma}^{(\beta)}(z)$ and its

Mellin transform were also investigated in [5]. These results we applied in [5] and [2] to study the mapping properties of the integral trasform with $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ as the kernel in spaces of tested and generalized functions $\mathcal{F}_{p,\mu}$ and $\mathcal{F}'_{p,\mu}$ by McBride[11] and in the space $\mathcal{L}_{\nu,r}$ (see, for example, [12]), respectively.

The present paper is devoted to investigate further properties of the function $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ in (1.1). Sections 2 and 3 deal with the special cases of this function when it is a function of hypergeometric and modified Besel type, respectively. Representations of $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ by fractional integrals of elementary functions are studied in Section 4. Recurrent relations for (1.1) are proved in Section 5. In Sections 6 and 7 the results obtained apply to solve in closed form certain differential equations of fractional order and Abel-Volterra integral equations of the third kind, respectively. Solvability of these equations in spaces of bounded functions is discussed in Section 8. We also note that the explicit solutions of another differential equations and integral equations of the third kind in terms of the Bessel-type function, defined by Kratzel[10], were given in our papers [6] and [1].

2. $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ AS A FUNCTION OF HYPERGEOMETRIC TYPE

In this section we consider the special cases of $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ when this function is expressed in terms of the Tricomi confluent hypergeometric $\Psi(a, c; z)$ defined for $a \in C$ ($Re(a) > 0$), $c \in C$ and $z \in C$ ($Re(z) > 0$) by [3, 6.5(2)]

$$\Psi(a, c; z) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} (1+t)^{c-a-1} e^{-zt}\, dt. \qquad \text{... (2.1)}$$

Lemma 1 — If $\gamma \in \mathbb{C}$, $Re(\gamma) > 0$, $\sigma \in \mathbb{R}$ and $z \in \mathbb{C}$ $Re(z) > 0$, then

$$\lambda^{(1)}_{\gamma,\sigma}(z) = e^{-z}\, \Psi(\gamma, \gamma+\sigma+1; z). \qquad \text{... (2.2)}$$

In particular, if $\sigma = m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$,

$$\lambda^{(1)}_{\gamma,m}(z) = z^{-\gamma} e^{-z} \sum_{k=0}^{m} \frac{(-m)_k (\gamma)_k}{k!} \frac{(-1)^k}{z^k}, \qquad \text{... (2.3)}$$

where $(\gamma)_k$ is the Pochhammer symbol

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1) \ldots (\gamma+k-1) = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)} \quad (k = 1, 2, ...), \qquad \text{... (2.4)}$$

and

$$\lambda^{(1)}_{\gamma,0}(z) = z^{-\gamma} e^{-z}. \qquad \text{... (2.5)}$$

PROOF : Let $z = x > 0$. Making the change of variable $\tau = t-1$ in (1.1) with $\beta = 1$ and using (2.1) with $a = \gamma$ and $c = \gamma+\sigma+1$, we have

$$\lambda^{(1)}_{\gamma,\sigma}(x) = \frac{e^{-x}}{\Gamma(\gamma)} \int_0^\infty \tau^{\gamma-1} (1+\tau)^{\sigma} e^{-xt}\, dt = \Psi(\gamma, \gamma+\sigma+1; x). \qquad \text{... (2.6)}$$

When $\sigma = m \in \mathbb{N} = \{1, 2, ...\}$, applying the binomial formula to (2.5), changing the variable $t = x\tau$ and using the definition of Gamma-function [3, 1.1(1)] and (2.4), we obtain

$$\lambda^{(1)}_{\gamma,m}(x) = \frac{e^{-x}}{\Gamma(\gamma)} \sum_{k=0}^{m} \frac{m(m-1)\dots(m-k+1)}{k!} \int_0^\infty t^{\gamma+k-1} e^{-xt}\, dt$$

$$= x^{-\gamma} e^{-x} \sum_{k=0}^{m} \frac{(-1)^k (-m)(-m+1)\dots(-m+k-1)}{k!} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)} x^{-k}$$

$$= x^{-\gamma} e^{-x} \sum_{k=0}^{m} \frac{(-m)_k (\gamma)_k}{k!} \frac{(-1)^k}{x^k}.$$

Thus (2.2) and (2.3) are proved for real $z = x$. The case of complex z, $Re(z) > 0$ follows by analytic continuation, and Lemma is proved.

When $a = 1$, then $\Psi(1, c; z)$ in (2.1) is expressed *via* the incomplete gamma-function $\Gamma(a, z)$ defined for $a \in \mathbb{C}$ and $z \in \mathbb{C}$, $Re(z) > 0$ by [3, 6.9(21)]

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-zt}\, dt. \qquad \text{... (2.7)}$$

Lemma 2 — if $c \in \mathbb{C}$ and $z \in \mathbb{C}$, $Re(z) > 0$, then

$$\Psi(1, c; z) = z^{1-c} e^{z} \Gamma(c-1, z). \qquad \text{... (2.8)}$$

PROOF : If $z = x$ real, then using (2.1) and make the change of variable $s = x(1 + t)$, we have

$$\Psi(1, c; x) = \int_0^\infty (1+t)^{c-2} e^{-xt}\, dt$$

$$= x^{1-\gamma} e^{x} \int_x^\infty s^{c-2} e^{-xs}\, ds = x^{1-c} e^{x} \Gamma(c-1, x),$$

and thus (2.8) is proved. For complex $z \in \mathbb{C}$, $Re(z) > 0$, this result is true by analytic continuation.

The following assertion characterize $\lambda^{1/\gamma}_{\gamma,\sigma}(z)$ in terms of the incomplete gamma-function (2.7).

Lemma 3 — If $\gamma \in \mathbb{C}$, $Re(\gamma) > 0$, $\sigma \in \mathbb{R}$ and $z \in \mathbb{C}$, $Re(z) > 0$, then

$$\lambda^{1/\gamma}_{\gamma,\sigma}(z) = \frac{1}{\gamma} e^{-z} \Psi(1, \sigma+2; z) = \frac{1}{\gamma} z^{-1-\sigma} \Gamma(\sigma+1, z). \qquad \text{... (2.9)}$$

In particular, if $\sigma = m \in \mathbb{N} = \{1, 2, ...\}$,

$$\lambda_{\gamma,\sigma}^{1/\gamma}(z) = \frac{e^{-z}}{\gamma^z} \sum_{k=0}^{m} (-m)_k \frac{(-1)^k}{z^k}, \qquad ... (2.10)$$

PROOF : For real $z = x$ the first result in (2.9) and (2.10) are proved similarly to the relations (2.2) and (2.3) in Lemma 1. The second result in (2.9) follows from the first one and (2.8).

3. $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ AS A FUNCTION OF MODIFIED BESSEL TYPE

In this section we present the special case of $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ when this function is expressed in terms of the modified Bessel function of the third kind $K_{-\gamma}(z)$ [4, Section 7.2.2]. According to [4, 7.12(19)] this function has the integral representation of the form

$$K_{-\gamma}(z) = \frac{\pi^{1/2}}{\Gamma(\gamma + 1/2)} \left(\frac{z}{2}\right)^{\gamma} \int_1^{\infty} (t^2 - 1)^{\gamma - 1/2} e^{-zt}\, dt \qquad ... (3.1)$$

for $z \in \mathbb{C}$ $(Re(z) > 0)$ and $\gamma \in C$ $(Re(\gamma) > -1/2)$.

Lemma 4 — If $\gamma \in \mathbb{C}$, $Re(\gamma) > -1/2$ and $z \in \mathbb{C}$, $Re(z) > 0$, then

$$\lambda_{\gamma,0}^{(2)}(z) = 2^{\gamma+1} \pi^{-1/2} z^{-\gamma} K_{-\gamma}(z) \qquad ... (3.2)$$

and

$$\lambda_{\gamma,1}^{(2)}(z) = \frac{z}{2} \lambda_{\gamma+1,0}^{(2)}(z) = 2^{\gamma+1} \pi^{-1/2} z^{-\gamma} K_{-(\gamma+1)}(z) \qquad ... (3.3)$$

PROOF : The result in (3.2), mentioned in (1.4), follows from (1.1) and (3.1). To prove the first relation in (3.3) we use the integration by parts. According to (1.1) and the relation

$$\Gamma(z+1) = z\Gamma(z) \qquad ... (3.4)$$

for the Gamma-function [3, 1.2(1)], we have

$$\lambda_{\gamma,1}^{(2)}(z) = \frac{2}{\Gamma(\gamma + 1/2)} \int_1^{\infty} (t^2 - 1)^{\gamma - 1/2} t e^{-zt}\, dt$$

$$= \frac{1}{(\gamma + 1/2)\,(\Gamma(\gamma + 1/2))} \int_1^{\infty} e^{-zt}\, d[(t^2 - 1)^{\gamma + 1/2}]$$

$$= \frac{z}{\Gamma(\gamma + 3/2)} \int_1^{\infty} (t^2 - 1)^{\gamma + 1/2} e^{-zt}\, dt = \frac{z}{2} \lambda_{\gamma+1,0}^{(2)}(z).$$

Thus the first relation in (3.3) is proved. The second one follows from (3.2).

Now we give the recurrent relation for $\lambda^{(2)}_{\gamma,m}(z)$ for $m \geq 2$.

***Theorem* 1** — *Let $\gamma \in \mathbb{C}$, $Re(\gamma) > -1/2$, $z \in \mathbb{C}$, $Re(z) > 0$, and $m \in \mathbb{N}$, $m \geq 2$. Then*

$$\lambda^{(2)}_{\gamma,m}(z) = \frac{1}{2}\left[z\lambda^{(2)}_{\gamma+1,m-1}(z) - (m-1)\,\lambda^{(2)}_{\gamma+1,m-2}(z)\right]. \qquad \text{... (3.5)}$$

PROOF : As in Lemma 4, applying the integration by parts and using (3.4) we have

$$\lambda^{(2)}_{\gamma,m}(z) = \frac{2}{\Gamma(\gamma+1/2)} \int_1^\infty (t^2-1)^{\gamma-1/2}\, t^m e^{-zt}\, dt$$

$$= \frac{1}{(\gamma+1/2)\,(\Gamma(\gamma+1/2))} \int_1^\infty [t^{m-1} e^{-zt}]\, d[(t^2-1)^{\gamma+1/2}]$$

$$= \frac{1}{\Gamma(\gamma+3/2)} \int_1^\infty (t^2-1)^{\gamma+1/2}\, [zt^{m-1} - (m-1)\, t^{m-2}]\, e^{-zt}\, dt$$

$$= \frac{1}{2}\,[z\lambda^{(2)}_{\gamma+1,m-1}(z) - (m-1)\,\lambda^{(2)}_{\gamma+1,m-2}(z)].$$

Thus (3.5) is proved, which completes the proof of theorem.

Corollary 1.1 — If $\gamma \in \mathbb{C}$, $Re(\gamma) > -1/2$ and $z \in \mathbb{C}$, $Re(z) > 0$, then

$$\lambda^{(2)}_{\gamma,2}(z) = 2^{\gamma+1}\,\pi^{-1/2}\, z^{(-\gamma+1)}\,[zK_{-(\gamma+2)}(z) - K_{-(\gamma+1)}(z)]. \qquad \text{... (3.6)}$$

and

$$\lambda^{(2)}_{\gamma,3}(z) = 2^{\gamma+1}\,\pi^{-1/2}\, z^{(-\gamma+1)}\,[zK_{-(\gamma+3)}(z) - 3K_{-(\gamma+2)}(z)]. \qquad \text{... (3.7)}$$

Corollary 1.1 follows from (3.5), (3.2) and (3.3).

4. $\lambda^{(\beta)}_{\gamma,\sigma}(x)$ AS FRACTIONAL INTEGRALS OF ELEMENTARY FUNCTIONS

In this section we show that the Bessel-type function (1.1) with real positive $z = x$ is representable as fractional integral of elementary functions. First we give its representation via the Erdelyi-Kober fractional integration operator $I^{\alpha}_{-;\beta;r}f$ defined for $\alpha \in \mathbb{C}$ $Re(\alpha) > 0)$, $\beta > 0$ and $\eta \in \mathbb{C}$ by [13, (18.7)]

$$I^{\alpha}_{-;\beta,\eta} f(x) = \frac{\beta x^{\beta\eta}}{\Gamma(\alpha)} \int_x^\infty \frac{t^{\beta(1-\alpha-\eta)-1} f(t)\, dt}{(t^\beta - x^\beta)^{1-\alpha}} \; (x \in \mathbb{R}_+ \equiv (0,+\infty)). \qquad \text{... (4.1)}$$

This generalized fractional integral is connected with the Liouville fractional integral $I_-^{\alpha} f$ in (1.8) by the following result.

Lemma 5 — Let $\alpha \in \mathbb{C}, Re(\alpha) > 0, \ \beta > 0$ and $\eta \in \mathbb{C}$. Then

$$(I^{\alpha}_{-;\beta,\eta} f)(x^{1/\beta}) = x^{r} (I^{\alpha}_{-} (t^{-\alpha-\eta} f(t^{1/\beta})))(x). \qquad \text{... (4.2)}$$

PROOF : Replacing x by $x^{1/\beta}$ in (4.1) and making the change of variable $t = s^{1/\beta}$, we have

$$(I^{\alpha}_{-;\beta,\eta} f)(x^{1/\beta}) = \frac{\beta x^2}{\Gamma(\alpha)} \int_{x^{1/\beta}}^{\infty} \frac{t^{\beta(1-\alpha-\eta)-1} f(t)\, dt}{(t^{\beta} - x)^{1-\alpha}}$$

$$= \frac{x^{\eta}}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{s^{-\alpha-\eta} f(s^{1/\beta})\, ds}{(s-x)^{1-\alpha}},$$

and obtain (4.2) in accordance with (1.8).

Now we prove that the modified Bessel-type function (1.1) is representable by the Erdelyi-Kober fractional integration operator (4.1) of exponential function.

***Theorem* 2** — *If* $\beta > 0, \ \gamma \in \mathbb{C}, \mathrm{Re}(\gamma) > (1/\beta) - 1$ *and* $\sigma \in \mathbb{R}$, *then*

$$\lambda^{(\beta)}_{\gamma,\sigma}(x) = (I^{\gamma+1-1/\beta}_{-;\beta,-(\gamma+\sigma/\beta)} e^{-t})(x) \ (x \in \mathbb{R}_+). \qquad \text{... (4.3)}$$

PROOF : Taking $z = x \in \mathbb{R}_+$ in (1.1) and making the change of variable $s = xt$, we have

$$\lambda^{(\beta)}_{\gamma,\sigma}(x) = \frac{\beta}{\Gamma(\gamma+1-1/\beta)} \int_{x}^{\beta} \left(\left(\frac{s}{x}\right)^{\beta} - 1 \right)^{\gamma-1/\beta} \left(\frac{s}{x}\right)^{\sigma} e^{-s} \frac{ds}{x}$$

$$= \frac{\beta x^{-(\beta\gamma+\sigma)}}{\Gamma(\gamma+1-1/\beta)} \int_{x}^{\infty} (s^{\beta} - x^{\beta})^{\gamma-1/\beta} s^{\sigma} e^{-s}\, ds.$$

(4.3) follows from here if we take into account (4.1) with $\alpha = \gamma + 1 - 1/\beta$ and $\eta = -(\gamma + \sigma/\beta)$.

Another representation for the modified Bessel-type function (1.1) is given via the Liouville fractional integral of power exponential function.

***Theorem* 3** — *If* $\beta > 0, \gamma \in \mathbb{C}, \mathrm{Re}(\gamma) > (1/\beta) - 1$ *and* $\sigma \in \mathbb{R}$, *then*

$$\lambda^{(\beta)}_{\gamma,\sigma}(x) = x^{-(\beta\gamma+\sigma)} (I_-^{\gamma+1-1/\beta} (t^{(\sigma+1)/\beta-1} \exp(-t^{1/\beta})))(x^{\beta}) \ (x \ \mathbb{R}_+). \qquad \text{... (4.4)}$$

PROOF : Applying (4.2) and (4.3), with x being replaced by x^{β} and $\alpha = \gamma + 1 - 1/\beta$ and $\eta = -(\gamma + \sigma/\beta)$ and $f(s) = e^{-s}$, we have

$$\lambda^{(\beta)}_{\gamma,\sigma}(x) = (I^{\gamma+1-1/\beta}_{-;\beta,-(\gamma+\sigma/\beta)}\, e^{-t})(x)$$

$$= x^{-(\beta\gamma+\sigma)}\,(I^{\gamma+1-1/\beta}_{-}\,(t^{(\sigma+1)/\beta-1}\exp(-t^{1/\beta})))\,(x^{\beta}),$$

and thus (4.4) is proved.

5. Recurrent Relations for $\lambda^{(\beta)}_{\gamma,\sigma}(z)$

In this section we prove three recurrent relations for the modified Besel-type function (1.1). The first two formulas connect three such functions.

***Theorem* 4** — *Let* $\beta > 0, \gamma \in \mathbb{C}, \mathrm{Re}(\gamma) > (1/\beta) - 1$ *and* $\sigma \in \mathbb{R}$. *Then for* $z \in \mathbb{C}$, $Re(z) > 0$,

$$\lambda^{(\beta)}_{\gamma,\sigma}(z) = \frac{z}{\beta}\lambda^{(\beta)}_{\gamma+1,\sigma+1-\beta}(z) - \frac{\sigma+1-\beta}{\beta}\lambda^{(\beta)}_{\gamma+1,\sigma-\beta}(z) \qquad \text{... (5.1)}$$

and

$$\lambda^{(\beta)}_{\gamma+1,\sigma}(z) = \frac{1}{\gamma+1-1/\beta}\left[\lambda^{(\beta)}_{\gamma,\sigma+\beta}(z) - \lambda^{(\beta)}_{\gamma,\sigma}(z)\right]. \qquad \text{... (5.2)}$$

Proof : Applying (1.1) and (3.4), integration by parts and using clear formula

$$(t^{\beta}-1)^{\gamma+1-1/\beta}\,t^{\sigma+1-\beta}\,e^{-zt}\,\big|_{1}^{\infty} = 0,$$

we have

$$\lambda^{(\beta)}_{\gamma,\sigma}(z) = \frac{1}{\Gamma(\gamma+2-1/\beta)}\int_{1}^{\infty} t^{\sigma+1-\beta}\,e^{-zt}\,d[(t^{\beta}-1)^{\gamma+1-1/\beta}]$$

$$= \frac{1}{\Gamma(\gamma+2-1/\beta)}$$

$$\times\left[(t^{\beta}-1)^{\gamma+1-1/\beta}\,t^{\sigma+1-\beta}\,e^{-zt}\,\big|_{1}^{\infty} - \int_{1}^{\infty}(t^{\beta}-1)^{\gamma+1-1/\beta}\,d[t^{\sigma+1-\beta}\,e^{-zt}]\right]$$

$$= \frac{z}{\Gamma(\gamma+2-1/\beta)}\int_{1}^{\infty}(t^{\beta}-1)^{\gamma+1-1/\beta}\,t^{\sigma+1-\beta}\,e^{-zt}\,dt$$

$$- \frac{\sigma+1-\beta}{\Gamma(\gamma+2-1/\beta)}\int_{1}^{\infty}(t^{\beta}-1)^{\gamma+1-1/\beta}\,t^{\sigma-\beta}\,e^{-zt}\,dt,$$

and thus obtain (5.1) in accordance with (1.1). Further,

$$\lambda^{(\beta)}_{\gamma+1,\sigma}(z)=\frac{\beta}{(\gamma+1-1/\beta)\,\Gamma(\gamma+1-1/\beta)}\int_1^\infty (t^\beta-1)^{\gamma-1/\beta}(t^\beta-1)\,t^\sigma e^{-zt}\,dt$$

$$=\frac{1}{\gamma+1-1/\beta}\left[\lambda^{(\beta)}_{\gamma,\sigma+\beta}(z)-\lambda^{(\beta)}_{\gamma,\sigma}(z)\right],$$

which completes the proof.

Corollary 4.1 — If $a\in\mathbb{C}, Re(a)>0, c\in\mathbb{C}$ and $z\in\mathbb{C}$, $Re(z)>0$, then

$$\Psi(a,c;z)=z\Psi(a+1,c+1;z)+(1+a-c)\,\Psi(a+1,c;z) \qquad \text{... (5.3)}$$

and

$$a\Psi(a+1,c+1;z)=\Psi(a+1,c;z)-\Psi(a,c;z). \qquad \text{... (5.4)}$$

PROOF : Setting $\beta=1$ on (5.1) and using (2.2), we have

$$\Psi(\gamma,\gamma+\sigma+1;z)=z\Psi(\gamma+1,\gamma+\sigma+2;z)-\sigma\Psi(\gamma+1,\gamma+\sigma+1;z).$$

Replacing γ by a and $\gamma+\sigma+1$ by c, we obtain (5.3). (5.4) is proved similarly.

Remark 1 : For $\beta=2$ and $\sigma=2$ (5.1) is the identity, since according to (3.2), (3.3) and (3.6) the both sides of (5.1) are equal to

$$2^{\gamma+1}\pi^{-1/2}z^{-(\gamma+1)}[zK_{-(\gamma+2)}(z)-K_{-(\gamma+1)}(z)].$$

If $\beta=2$ and $\sigma=0$, then in accordance with (3.2), (3.3) and (3.6), (5.2) is reduced to the relation

$$2(\gamma+1)\,K_{-(\gamma+1)}(z)=z[K_{-(\gamma+2)}(z)-K_{-\gamma}(z)],$$

that coincides with the known result in [4, 7.11(25)], if we take into account the relation [4, 7.2(14)]

$$K_{-\gamma}(z)=K_\gamma(z). \qquad \text{... (5.5)}$$

Now we prove the relation connected four functions of the form (1.1) with the same γ and different σ.

Theorem 5 — *Let* $\beta>0, \gamma\in\mathbb{C}, \mathrm{Re}(\gamma)>(1/\beta)-1$ *and* $\sigma\in\mathbb{R}$. *Then for* $z\in\mathbb{C}$, $Re(z)>0$,

$$(\gamma\beta+\sigma)\,\lambda^{(\beta)}_{\gamma,\sigma}(z)=z\,[\lambda^{(\beta)}_{\gamma,\sigma+1}(z)-\lambda^{(\beta)}_{\gamma,\sigma+1-\beta}(z)]+(\sigma+1-\beta)\,\lambda^{(\beta)}_{\gamma,\sigma-\beta}(z). \qquad \text{... (5.6)}$$

PROOF : According to (5.2).

$$\lambda^{(\beta)}_{\gamma+1,\sigma+1-\beta}(z)=\frac{1}{\gamma+1-1/\beta}\left[\lambda^{(\beta)}_{\gamma,\sigma+1}(z)-\lambda^{(\beta)}_{\gamma,\sigma+1-\beta}(z)\right]$$

and

$$\lambda^{(\beta)}_{\gamma+1,\sigma-\beta}(z)=\frac{1}{\gamma+1-1/\beta}\left[\lambda^{(\beta)}_{\gamma,\sigma}(z)-\lambda^{(\beta)}_{\gamma,\sigma-\beta}(z)\right].$$

Substituting these relations into (5.1), we obtain (5.6).

Corollary 5.1 – If $a \in \mathbb{C}, Re(a) > 0, c \in \mathbb{C}$ and $z \in \mathbb{C}, Re(z) > 0$, then

$$(c+z)\, \Psi(a, c+1; z) = z\Psi(a, c+2; z) + (c-a)\, \Psi(a, c; z). \qquad \dots (5.7)$$

PROOF : Setting $\beta = 1$ in (5.6) and using (2.2), we have

$$(\gamma+\sigma+z)\, \Psi(\gamma, \gamma+\sigma+1; z) = z\Psi(\gamma, \gamma+\sigma+2; z) + \sigma\Psi(\gamma, \gamma+\sigma; z),$$

that coincides with (5.7) for $a = \gamma$ and $c = \gamma+\sigma$.

Corollary 5.2 — If $\gamma \in \mathbb{C}, Re(\gamma) > -1/2$ and $z \in \mathbb{C}, Re(z) > 0$, then

$$(2\gamma+5)\, K_{-(\gamma+2)}(z) + zK_{-\gamma}(z) = z^2 K_{-(\gamma+3)} + (2\gamma+2-z)\, K_{-(\gamma+1)}(z). \qquad \dots (5.8)$$

PROOF : (5.8) follows from (5.6) if we take $\beta = \sigma$ and use the relations (3.2), (3.3), (3.6) and (3.7).

6. $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ AS A SOLUTION OF DIFFERENTIAL EQUATION OF FRACTIONAL ORDER

For $\alpha \geq 0$ let D^{α}_{-} be the fractional differentiation operator in (1.9) for $\alpha > 0$ and the identical operator $D^0_{-} \equiv I$ for $\alpha = 0$. For $\alpha \geq 0$, $\beta > 0$, $\gamma \in \mathbb{C}$ and $\sigma \in \mathbb{R}$ we denote by $L^{\alpha,\beta}_{\gamma,\sigma}$ the fractional differentiation operator defined by

$$L^{\alpha,\beta}_{\gamma,\sigma} = x(D^{\alpha+\beta+1}_{-} - D^{\alpha+1}_{-}) + (\sigma+\alpha+1-\beta)\, D^{\alpha}_{-}. \qquad \dots (6.1)$$

Theorem 6 — *Let $\alpha \geq 0$, $\beta > 0$, $\gamma \in \mathbb{C}$, $Re(\gamma) > (1/\beta) - 1$ and $\sigma \in \mathbb{R}$. Then for $x > 0$ the relation*

$$(L^{\alpha,\beta}_{\gamma,\sigma}\, \lambda^{(\beta)}_{\gamma,\sigma-\beta})\,(x) = (\gamma\beta+\sigma+\alpha)\,(D^{\alpha+\beta}_{-}\, \lambda^{(\beta)}_{\gamma,\sigma-\beta})\,(x) \qquad \dots (6.2)$$

holds for the modifed Bessel-type function $\lambda^{(\beta)}_{\gamma,\sigma-\beta}(x)$.

PROOF : Applying (1.7) with α being replaced by $\alpha+\beta$ to $\lambda^{(\beta)}_{\gamma,\sigma-\beta}(x)$ and using (5.6) with σ being replaced by $\sigma+\alpha$, we have for $x > 0$,

$$(\gamma\beta+\sigma+\alpha)\,(D^{\alpha+\beta}_{-}\, \lambda^{(\beta)}_{\gamma,\sigma-\beta})\,(x) = (\gamma\beta+\sigma+\alpha)\; \lambda^{(\beta)}_{\gamma,\sigma+\beta})\,(x)$$

$$= x[\lambda_{\gamma,\sigma+\alpha+1}(x) - \lambda^{(\beta)}_{\gamma},\sigma+\alpha+1-\beta(x)] + (\sigma+\alpha+1-\beta)\; \lambda^{(\beta)}_{\gamma},\sigma+\alpha-\beta(x).$$

Applying (1.7) again, we obtain

$$(\gamma\beta+\sigma+\alpha)\,(D^{\alpha+\beta}_{-}\, \lambda^{(\beta)}_{\gamma,\sigma-\beta})\,(x)$$

$$= x\,[(D_-^{\alpha+\beta}\lambda_{\gamma,\sigma-\beta}^{(\beta)})(x) - (D_-^{\alpha+\beta}\lambda_{\gamma,\sigma-\beta}^{(\beta)})(x)] + (\sigma+\alpha+1-\beta)((D_-^{\alpha}\lambda_{\gamma,\sigma-\beta}^{(\beta)})(x),$$

which yields (6.2) in accordance with (6.1), and theorem is proved.

Corollary 6.1 — If $\alpha \geq 0, \beta > 0, \gamma \in \mathbb{C}, Re(\gamma) > (1/\beta) - 1$ and $\sigma \in \mathbb{R}$, then the modified Bessel-type function

$$y(x) = \lambda_{\gamma,\sigma-\beta}^{(\beta)}(x) \qquad \text{... (6.3)}$$

is a solution of the differential equation of fractional order

$$x[(D_-^{\alpha+\beta+1}y)(x) - (D_-^{\alpha+1}y)(x)] - (\gamma\beta+\sigma+\alpha)(D_-^{\alpha+\beta}y)(x)$$

$$+ (\sigma+\alpha+1-\beta)(D_-^{\alpha}y)(x) = 0 \; (x > 0), \qquad \text{... (6.4)}$$

in particular, of the differential equation

$$x(D_-^{\beta+1}y)(x) - (\gamma\beta+\sigma)(D_-^{\beta}y)(x) - xy'(x) + (\sigma+1-\beta)\,y(x) = 0 \; (x > 0). \qquad \text{... (6.5)}$$

Corollary 6.2 — If $\alpha \geq 0, \gamma \in \mathbb{C}, Re(\gamma) > 0$ and $\sigma \in \boldsymbol{R}$, then the differential equation of fractional order

$$x(D_-^{\alpha+2}y)(x) - (\gamma+\sigma+\alpha+x)(D_-^{\alpha+1}y)(x) + (\sigma+\alpha)(D_-^{\alpha}y)(x) = 0 \; (x > 0). \qquad \text{... (6.6)}$$

has the solution

$$y(x) = \lambda_{\gamma,\sigma-1}^{(1)}(x) = e^{-x}\,\Psi(\gamma, \gamma+\sigma; x), \qquad \text{... (6.7)}$$

where $\Psi(\gamma, \gamma+\sigma, x)$ is the Tricomi confluent hypergeometric function (2.1).

Corollary 6.3 — If $\alpha \geq 0, \gamma \in \mathbb{C}, Re(\gamma) > 0$ and $m = 0, 1, 2, \ldots$, then the power exponential function

$$y(x) = \lambda_{\gamma,m}^{(1)}(x) = x^{-\gamma}e^{-x}\sum_{k=0}^{m}\frac{(-m)_k(\gamma)_k}{k!}\frac{(-1)^k}{x^k} \qquad \text{... (6.8)}$$

is the solution of the differential equation of fractional order

$$x(D_-^{\alpha+2}y)(x) - (\gamma+m+1+\alpha+x)(D_-^{\alpha+1}y)(x)$$

$$+ (m+1+\alpha)(D_-^{\alpha}y)(x) = 0 \; (x > 0). \qquad \text{... (6.9)}$$

In particular,

$$y(x) = x^{-\gamma}e^{-x}. \qquad \text{... (6.10)}$$

is the solution of the equation

$$x(D_{-}^{\alpha+2} y)(x) - (\gamma + 1 + \alpha + x)(D_{-}^{\alpha+1} y)(x) + (1+\alpha)(D_{-}^{\alpha} y)(x) = 0 \ (x > 0). \quad \dots (6.11)$$

Corollary 6.4 — If $\alpha \geq 0$, $\gamma \in \mathbb{C}$, $Re(\gamma) > -1/2$ and $m = 2, 3, \dots$, then the differential equation of fractional order

$$x(D_{-}^{\alpha+3} y)(x) - (2\gamma + m + \alpha + 1)(D_{-}^{\alpha+2} y)(x) - x(D_{-}^{\alpha+1} y)(x)$$

$$+ (m+\alpha)(D_{-}^{\alpha} y)(x) = 0 \ (x > 0) \quad \dots (6.12)$$

has the solution

$$y(x) = \lambda_{\gamma, m}^{(2)}(x), \quad \dots (6.13)$$

where $\lambda_{\gamma, m}^{(2)}(x)$ is given by recurrent relation (3.5).

In particular, the equations

$$x(D_{-}^{\alpha+3} y)(x) - (2\gamma + 2 + \alpha)(D_{-}^{\alpha+2} y)(x)$$

$$- x(D_{-}^{\alpha+1}(x) + (1+\alpha)(D_{-}^{\alpha} y)(x) = 0 \ (x > 0) \quad \dots (6.14)$$

and

$$x(D_{-}^{\alpha+3} y)(x) - (2\gamma + 3 + \alpha)(D_{-}^{\alpha+2} y)(x)$$

$$- x(D_{-}^{\alpha+1} y)(x) + (2+\alpha)(D_{-}^{\alpha} y)(x) = 0 \ (x > 0) \quad \dots (6.15)$$

have the solutions

$$y(x) = x^{-\gamma} K_{-\gamma}(x) \quad \dots (6.16)$$

and

$$y(x) = x^{-\gamma} K_{(-\gamma+1)}(x), \quad \dots (6.17)$$

respectively.

PROOF : Corollary 6.1 follows from Theorem 6. According to (2.2) and (3.5), Corollaries 6.2 and 6.4 are particular cases of Corollary 6.1, when $\beta = 1$ and $\beta = 2, \sigma = m+1$, respectively. By (2.3), Corollary 6.3 follows from Corollary 6.2 for $\sigma = m+1$.

Remark 2 : The results in (6.3), (6.7), (6.8), (6.10), (6.13), (6.16) and (6.17) give the solutions of the corresponding homogeneous ordinary differential equations, obtained from (6.4), (6.5), (6.6), (6.9), (6.11), (6.12), (6.14) and (6.15) for $\alpha = m = 0, 1, 2, \dots$ and $\beta = m = 1, 2 \dots$. For example, the function $y(x) = x^{-\gamma} e^{-x}$ in (6.10) is a solution of the ordinary differential equation

$$xy''(x) + (\gamma + 1 + x)y'(x) + y(x) = 0, \quad \dots (6.18)$$

being the equation of the form (6.11) with $\alpha = 0$. Such a result can be also proved by direct calculations.

7. $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ AS A SOLUTION OF AN INTEGRAL EQUATION

In the previous section we showed that the recurrent relations (5.6) together with the fractional differentiation formula (1.7) for the modified Bessel-type function (1.1) can be applied to solution in closed form of certain differential equations of fractional order, in particular for the ordinary differential equations. Here we show that the fractional integration formula (1.6) together with (5.6) can be used to solution in closed form of certain integral equations.

Theorem 7 — *Let $\beta > 0$, $\gamma \in \mathbb{C}$, $Re(\gamma) > (1/\beta) - 1$ and $\sigma \in \mathbb{R}$. Then the modified Bessel-type function*

$$\varphi(x) = \lambda^{(\beta)}_{\gamma,\sigma+1}(x) \qquad \text{... (7.1)}$$

is a solution of the homogeneous integral equation of the third kind

$$x\varphi(x) = \int_x^\infty \left[\gamma\beta + \sigma + \frac{(t-x)^{\beta-1}}{\Gamma(\beta)} x + (\beta - \sigma - 1) \frac{(t-x)^{\beta}}{\Gamma(\beta+1)} \right] \varphi(t)\, dt \cdot (x > 0) \qquad \text{... (7.2)}$$

PROOF : Applying (1.6) with $\alpha = 1, \alpha = \beta$ and $\alpha = \beta + 1$ to $\lambda^{(\beta)}_{\gamma,\sigma}(x)$, $\lambda^{(\beta)}_{\gamma,\sigma+1-\beta}(x)$ and $\lambda^{(\beta)}_{\gamma,\sigma-\beta}(x)$, respectively and substitute the relations obtained to (5.6), we have for $x > 0$,

$$(\gamma\beta + \sigma)\,(I^1_- \lambda^{(\beta)}_{\gamma,\sigma+1})(x) = (\gamma\beta + \sigma)\,\lambda^{(\beta)}_{\gamma,\sigma})(x)$$

$$= x\,[\lambda^{(\beta)}_{\gamma,\sigma+1}(x) - \lambda^{(\beta)}_{\gamma,\sigma+1-\beta}(x)] + (\sigma + 1 - \beta)\,\lambda^{(\beta)}_{\gamma,\sigma-\beta}(x)$$

$$= x\left[\lambda^{(\beta)}_{\gamma,\sigma+1}(x) - (I^{\beta}_- \lambda^{(\beta)}_{\gamma,\sigma+1}(x) \right] + (\sigma + 1 - \beta)\,(I^{\beta+1}_- \lambda^{(\beta)}_{\gamma,\sigma+1})(x),$$

which according to (1.8) yields (7.2).

Corollary 7.1 — If $\gamma \in \mathbb{C}$, $Re\ (\gamma) > 0$ and $\sigma \in \boldsymbol{R}$, then the integral equation

$$x\varphi(x) = \int_x^\infty [\gamma + x + \sigma(1 + x - t)]\, \varphi(t)\, dt\, (x > 0) \qquad \text{... (7.3)}$$

has the solution

$$\varphi(x) = \lambda^{(1)}_{\gamma,\sigma+1}(x) = e^{-x}\, \Psi(\gamma, \gamma + \sigma; x), \qquad \text{... (7.4)}$$

where $\Psi(\gamma, \gamma + \sigma; x)$ is the Tricomi confluent hypergeometric function (2.1).

Corollary 7.2 — If $\gamma \in \mathbb{C}$, $\mathrm{Re}(\gamma) > 0$ and $m = 0, 1, 2, \ldots$, then the power exponential function

$$\varphi(x) = \lambda^{(1)}_{\gamma, m}(x) = x^{-\gamma} e^{-x} \sum_{k=0}^{m} \frac{(-m)_k (\gamma)_k}{k!} \frac{(-1)^k}{x^k} \quad \ldots (7.5)$$

is the solution of the integral equation

$$x\varphi(x) = \int_x^{\infty} [\gamma + x + (m-1)(1 + x - t)]\, \varphi(t)dt \; (x > 0). \quad \ldots (7.6)$$

In particular,

$$(\varphi)x = x^{-\gamma} e^{-x} \quad \ldots (7.7)$$

is the solution of the equation

$$x\varphi(x) = \int_x^{\infty} (\gamma + t - 1)\varphi(t)\, dt \quad (x > 0). \quad \ldots (7.8)$$

Corollary 7.3 — If $\gamma \in \mathbb{C}$, $\mathrm{Re}(\gamma) > -1/2$ and $m = 2, 3, \ldots$, then the integral equation

$$x\varphi(x) + \int_x^{\infty} \left[2\gamma + m - 1 + (t - x) + \left(1 - \frac{m}{2}\right)(t - x)^2 \right] \varphi(t)dt \; (x > 0) \quad \ldots (7.9)$$

has the solution

$$y(x) = \lambda^{(2)}_{\gamma, m}(x), \quad \ldots (7.10)$$

where $\lambda^{(2)}_{\gamma, m}(x)$ is given by recurrent relation (3.5).

In particular, the equations

$$x\varphi(x) = \int_x^{\infty} [2\gamma - 1 + (t - x) + (t - x)^2]\, \varphi(t)dt \; (x > 0) \quad \ldots (7.11)$$

and

$$x\varphi(x) = \int_x^{\infty} \left[2\gamma + (t - x) + \frac{(t - x)^2}{2} \right] \varphi(t)dt \; (x > 0) \quad \ldots (7.12)$$

have the solutions

$$y(x) = x^{-\gamma} K_{-\gamma}(x) \quad \ldots (7.13)$$

and

$$y(x) = x^{-\gamma} K_{-(\gamma+1)}(x) \qquad \text{... (7.14)}$$

respectively.

PROOF : According to (2.2) and (3.5), Corollaries 7.1 and 7.3 are particular cases of Theorem 7, when $\beta = 1$ and $\beta = 2, \sigma = m-1$, respectively. By (2.3), Corollary 7.2 follows from Corollary 7.1 for $\sigma = m-1$.

Remark 3 : The relation (7.1) is the Volterra nonhomogeneous integral equation of the third kind having for $0 < \beta < 1$ the weak power singularity. The results in Theorem 7 and Corollaries 7.1-7.3 show that such equations, unlike Volterra integral equations of the second kind, have nontrivial solutions.

8. Solvability of Differential and Integral Equations in the Space of Bounded Functions

In sections 6 and 7 we proved that the modified Bessel-type of the third kind $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ is the solution of the certain homogeneous differential and integral equations of fractional order. Here we give the conditions for solvability these equations in a space of bounded functions. We denote by $B(\mathbb{R}_+)$ the space of functions bounded on $\mathbb{R}_+ = [0, \infty]$ and by $B_0(\mathbb{R}_+)$ its subspace consisting of function vanishing at infinity:

$$B_0(\mathbb{R}_+) = \left\{ f \in (\mathbb{R}_+) : \lim_{x \to +\infty} f(x) = 0 \right\}. \qquad \text{... (8.1)}$$

Lemma 6 — If $\beta > 0, \gamma \in \mathbb{C}$ and $\sigma \in \mathbb{R}$ be such that

$$\frac{1}{\beta} - 1 < \mathrm{Re}(\gamma) < -\frac{\sigma}{\beta}, \qquad \text{... (8.2)}$$

then $\lambda^{(\beta)}_{\gamma,\sigma}(x) \in B_0(\mathbb{R}_+)$.

PROOF : By analiticity $\lambda^{(\beta)}_{\gamma,\sigma}(z)$ is bounded on $\mathbb{R}_+ = (0, \infty)$. It is known [5] that the modified Bessel-type function (1.1) has the following asymptotic behavior near zero and infinity:

$$\lambda^{(\beta)}_{\gamma,\sigma}(z) \sim A, \; A = \frac{\Gamma(-\gamma - \sigma/\beta)}{\Gamma(1-(\sigma+1)/\beta)} \; (z \to 0) \qquad \text{... (8.3)}$$

provided that the condition in (8.2) are satisfied, and

$$\lambda^{(\beta)}_{\gamma,\sigma}(z) \sim Be^{-z} z^{1/\beta - \gamma - 1}, \; B = \beta^{\gamma + 1/\beta} \; (z \to \infty, \mathrm{Re}(z) > 0) \qquad \text{... (8.4)}$$

provided that $\mathrm{Re}(\gamma) > (1/\beta) - 1$. Then

$$\lim_{x \to +0} \lambda^{(\beta)}_{\gamma,\sigma}(x) = A, \qquad \lim_{x \to +\infty} \lambda^{(\beta)}_{\gamma,\sigma}(x) = 0,$$

and hence we obain the result in lemma.

From Theorem 6 and 7 we obtain the conditions for the solvability of the differential equations of fractional order (6.4) and of the Volterra integral equation (7.2) in the space $B_0(\mathbb{R}_+)$.

Theorem 8 — *Let $\alpha \geq 0, \beta > 0, \gamma \in \mathbb{C}$ and $\sigma \in \boldsymbol{R}$ be such that*

$$\frac{1}{\beta} - 1 < \mathrm{Re}(\gamma) < 1 - \frac{\sigma}{\beta}. \qquad \text{... (8.5)}$$

Then the differential equation of fractional order (6.4) is solvable in $B_0(\mathbb{R}_+)$ and its solution is given by (6.3).

Corollary 8.1. — If $\alpha \geq 0, \gamma \in \mathbb{C}$ and $\sigma \in \mathbb{R}$ be such that $0 < Re(\gamma) < 1 - \sigma$, then the differential equation of fractional order (6.6) is solvable in $B(\mathbb{R}_+)$ as its solution is given by (6.7).

Theorem 9 — *Let $\beta > 0, \gamma \in \mathbb{C}$ and $\sigma \in \mathbb{R}$ be such that*

$$\frac{1}{\beta} - 1 < \mathrm{Re}(\gamma) < -1 - \frac{\sigma}{\beta}. \qquad \text{... (8.7)}$$

Then the integral equation (7.2) is solvable in $B_0(\mathbb{R}_+)$ and its solution is given by (7.1).

Corollary 9 — If $\gamma \in C$ and $\sigma \in \mathbf{R}$ be such that

$$0 < \mathrm{Re}\ (\gamma) < -1 - \sigma, \qquad \text{... (8.8)}$$

then the integral equation (7.3) is solvable in $B_0(\mathbb{R}_+)$ and its solution is given by (7.4)

Remark 4 : The conditions in (8.5) and (8.7) can be satisfied only if $\sigma < 2\beta - 1$ and $\sigma < -2\beta - 1$, respectively. Therefore the solutions (6.8) and (6.9) of th differential equations (6.9) and (6.12) as well as the solution (7.5) and (7.10) of the integral equations (7.6) and (7.9) do not belong to the space $B_0(\mathbb{R}_+)$. They have singularities at z = 0. and therefore the equations above are solvable in another spaces. For example, if $0 < \mathrm{Re}(\gamma) < 1$, the equations (6.11) and (7.8) are solvable in the space of functions, bounded on $\mathbf{R}_+$ and vanishing at infinity and having integrable singularity at zero, and their solutions have the same form: (6.10) and (7.7).

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REFERENCES

1. B. Bonilla, A. A. Kilbas, M. Rivero, J. Rodriguez and J. J. Trujillo, *Spec. Funct.* **8** (1999), no. 1-2, to appear.
2. B. Bonilla, A. A. Kilbas, M. Rivero, Rodriguez and J. J. Trujillo, Bessel-type integral transform in $L_{\nu,p}$-space, Rev. Acad. Canaria Cienc., *to appear.*
3. A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi Higher Transcendental Functions, *Vol. I, McGraw-Hill, New York-Toronto-London, 1953.*
4. A. Erdeyi, W. Magnus, F. Oberhettinger and F. G. Tricomi Higher Transcendental Functions, *Vol. II, McGraw-Hill, New York-Toronto-London, 1953.*
5. H. J. Glaeske, A. A. Kilbas and M. Saigo, A modified Bessel-type integral transform and its compositions with fractional calculus operators on spaces $\mathcal{F}_{p,\mu}$ and $\mathcal{F}'_{p,\mu}$, *J. Comput. Anal. Math.*, to appear.

6. A. A. Kilbas, B. Bonilla, M. Rivero, . Rodriguez and J. J. Trujillo, *Dokl. Nat. Akad. Nauk Belarusi* **42** (1998), no. 2, 25-29.

7. E. Kratzel, *Wiss. Z. Friedrich-Schiller-Univ. Jena/Thuringen* **14** (1965), 369-381.

8. E. Kratzel, *Wizz. Z. Friedrich-Schiller-Univ. Jena/Thuringen* **14** (1965), 333-90.

9. E. Kratzel, *Math. Nachr.* **30** (1965), 327-34.

10. E. Kratzel, *Generalized functions and operational calculus (Proc. Conf., Varna,* 1975, 148-155, Bulg. Acad. Sci., Sofia, 1979.

11. A. C. McBride, *Fractional Calculus and Integral Transforms of Generalized Functions*, Res. Notes Math., **31**, Pitnam, San-Francisco-London-Melbourne, 1979.

12. P. G. Rooney, *Proc. Royal Soc. Edinburgh*, Sect. A **93** (1982/83), 265-97.

13. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.