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A *D/D/*1 QUEUEING PROCESS WITH VARYING SERVICE TIMES

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This paper discusses a deterministic single-server queueing model in which the interarrival time is fixed and the service times vary from one customer to another, although they are always less than the interarrival time. Explicit formulas are given for the number of customers in the system as a function of time, and the waiting time of each customer. Moreover, a necessary and sufficient condition is given for the number of customers in the system to remain at or above a given level.

1. INTRODUCTION

In [1], the following situation is examined. Initially, there are i ships parked in a harbor waiting to be unloaded. It takes the nth ship k_n units of time to get to the unloading dock, and b units of time to be unloaded. A new ship arrives every a units of time. The case in which $b+k_n < a$ for all n is examined briefly, but for most of the paper it is assumed that $b+k_n < a$ for all n. (Neither [1] nor the present paper deal with the general case in which $b+k_n \geq a$ holds for some natural numbers n and $b+k_n < a$ holds for other n.)

In [1] the assertion is made that having $b+k_n < a$ for all n entails that the queue must eventually become empty. The results of paper show that this assertion is erroneous.

In section 2, we fix the notation used in this paper. Then in the remainder of this paper, we give explicit formulas for the number of customers (ships) in the system as a function of time, the infimum of the number of customers, and the waiting time of the nth customer to be serviced as a function of n. We also give a necessary and sufficient condition for the number of customers in the system to remain at or above a given level. In particular, it is shown by example that the system might very well remain nonempty for all time.

2. ASSUMPTIONS AND NOTATION

We denote $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{W} = \{0, 1, 2, \ldots\}$. If t is a real number, we let $[t]$ denote the greatest integer of t, and $\{t\} = t - [t]$.

Let $a > 0$ and let $\{b_n\}_{n=1}^{\infty}$ be a sequence in which $0 < b_n < a$ for all $n \in \mathbb{N}$. As in [1], we consider a deterministic single-server queue in which there are initially i customers. We assume that a new customer arrives at the end of each successive time interval of length a. The service time of the nth customer to be services is b_n. (In the notation of [1], $b_n = b + k_n$). We assume that $b_n < a$.

We let t denote the amount of time that has elapsed. We define $N(t)$ to be the number of customers in the system at time t. Let denote $L = \liminf_{t \to \infty} N(t)$. Since there is at least one customer serviced between any two arrivals, $\limsup_{t \to \infty} N(t) = L + 1$.

If $k \in \{0, 1, ..., i - 1\}$, we define $T_k = \inf \{t \mid N(t) = k\}$.

We use the convention that $\inf \emptyset = +\infty$. We shall see below that $T_k = +\infty$ is quite possible for $k < i - 1$.

Clearly, $T_{i-1} = b_1$, and $T_{i-1} \leq T_{i-2} \leq \ldots \leq T_0$.

Moreover, if $T_k < +\infty$, then $T_{i-1} < T_{i-2} < \ldots < T_k$. If $T_k < +\infty$, then it is a moment in time when a service is completed. Therefore, for some $m_k \in I\!N$,

$$T_k = \sum_{j=1}^{m_k} b_j.$$

We shall use $W_q^{(n)}$ to denote the amount of time that the nth customer to be serviced must wait in the queue before being serviced.

We define $n_0 = \inf \{n \in I\!W \mid an > T_0\}$. We also define

$$B(t) = \sup \left\{ n \in I\!W \mid \sum_{j=1}^{n} b_j \leq t \right\},$$

where we employ the convention that an empty sum is 0. We denote $M(t) = i + [t/a] - B(t)$.

We also denote

$$S = \sum_{j=1}^{\infty} (1 - bj/a).$$

Of course, $S > 0$. It could be that $S = +\infty$.

3. Proposition

$$N(t) = \begin{cases} M(t) & \text{if } 0 \leq t \leq T_0 \\ 0 & \text{if } T_0 < t < an_0 \\ 1 & \text{if } T_0 < an \leq t < an + b_{n+i}, n \in I\!N \\ 0 & \text{if } T_0 < an + b_{n+i} \leq t < a(n+1), n\ a \in I\!N. \end{cases}$$

Of course, if $T_0 = +\infty$ then only the top line of the above formula applies.

PROOF : If $0 \leq t \leq T_0$, then the server has been busy during the whole time interval $[0, t]$, so $[t/a]$ is the number of arrivals by time t, and $B(t)$ is the number serviced. Hence, $N(t) = M(t)$.

Since $b_n < a$, at least one customer is serviced between any two arrivals. Therefore, if $N(s) = k$, then $N(t) \leq k+1$ for all $t \geq s$. By definition of T_0, $N(T_0) = 0$. If a customer arrives at time $t > T_0$, the number $N(t)$ of customers increases from 0 to 1. The arrival time must be t = an for some $n \in IN$, and the customer is the $(n + i)$th customer to be serviced. When the service has been performed after b_{n+i} units of time, the value of $N(t)$ returns to 0 until the next customer arrives.

■

4. PROPOSITION

Let $n \in IN$.

$$W_q^{(n)} = \begin{cases} \sum_{j=1}^{n-1} b_j - \min(0, (n-i)a) & \text{if } n < i + n_0 \\ 0 & \text{if } n \geq i + n_0 \end{cases}$$

PROOF : The arrival time of the nth customer is min (0, $(n - i)a$).

If min (0, $(n - i)\ a) > T_0$ then the there is no queue, so the waiting time is 0. This situation occurs if and only if $n - i > n_0$. By definition of T_0, it is impossible for T_0 to be an arrival time. If $0 \leq \min(0, (n-i)\ a) < T_0$, is equivalently $n \geq i + n_0$, the waiting time is equal to the time required to service the first $n - 1$ customers minus the arrival time. ■

5. *Lemma* — Let $t \geq 0$ and $k \in IW$. Then $M(t) \leq k$ if and only if

$$\sum_{j=1}^{i-k+[t/a]} (1 - b_j/a) \geq i - k - \{t/a\}.$$

PROOF : Each of the following assertions is plainly equivalent to its predecessor.

$M(t) \leq k$.

$B(t) \geq i - k + [t/a]$.

$$\sum_{j=1}^{i-k+[t/a]} b_j \leq t \quad .$$

$$\sum_{j=1}^{i-k+[t/a]} (1 - b_j/a) \geq i - k - \{t/a\}.$$

■

6. *Theorem* — *Let* $k \in \{0, \ldots, i-1\}$. *Then* $T_k < +\infty$ *iff.*

$S > i - k - 1$.

PROOF : Suppose that $T_k < +\infty$. Then $M(T_k) = k$. So using the lemma and the fact that T_k is not a multiple of a, we have

$$S \geq \sum_{j=1}^{i+[T_k/a]-k} (1 - b_j/a)$$

$$\geq i - k - \{T_k/a\}$$

$$> i - k - 1.$$

Suppose that $T_k = +\infty$. Then for all $t \geq 0, M(t) \geq k+1$. Using the lemma,

$$S = \lim_{t \to \infty} \sum_{j=1}^{i+[t/a]-k} (1 - b_j/a)$$

$$\leq \liminf_{t \to \infty} (i - k - \{t / a\})$$

$$= i - k - 1.$$ ■

7. *Example* — Let $i > 2$ and $k \in \{0, \ldots, i-2\}$. Suppose that $a = 1$ and $b_n = 1 - (i-k-1)/2^n$. Then

$$\sum_{j=1}^{\infty} (1 - b_j/a) = \sum_{j=1}^{\infty} (i-k-1)/2^j$$

$$= i - k - 1.$$

Hence, $T_k = +\infty$ and $T_{k+1} < +\infty$ by the theorem. So the number of customers starts at i, eventually gets down to $k+1$, and then fluctuates between $k+1$ and $k+2$ for ever more.

8. *Corollary* — Suppose that $S \leq i - 1$. Then $i - s - 1 < L \leq i - s$.

PROOF : Since $S \leq i-1$, Theorem 6 shows that $L > 0$. By definition of $L, T_L < +\infty$ and $T_{L-1} = +\infty$. Again by the theorem, $S > i - L - 1$, and $S \leq i - (L-1) - 1$. The result follows. ■

We remark in passing that since L is a whole number, L is uniquely determined by the above formula.

As we observed earlier, if $T_k < \infty$, then T_k can be written as a partial sum of the series $\sum b_j$, and m_k is defined to be the number of summands. The following result shows that m_k can be found by an examination of the partial sums of the series S.

9. *Proposition* — Let $k \in \{0, \ldots, i-1\}$, and suppose that $T_k < +\infty$. Then

$$m_k = \min \left\{ m \in \mathbb{N} \mid \sum_{j=1}^{m} (1 - b_j/a) > i - k - 1 \right\}.$$

PROOF : Since $T_k < T_0, N(T_k) = M(T_k) = k$. By definition, $B(T_k) = m_k$. So $m_k = i - k + [T_k/a]$. Thereofore,

$$\sum_{j=1}^{m_k} (1 - b_j/a) = m_k - (1/a) \sum_{j=1}^{m_k} b_j$$

$$= i - k - \{T_k/a\}$$

$$> i - k - 1.$$

It suffices to show that

$$\sum_{j=1}^{m_k - 1} (1 - b_j/a) \leq i - k - 1.$$

A straight forward computation reduces this inequality to the assertion that $b_k/a \leq \{T_k/a\}$.

To prove the latter inequality, we note that at time T_k the number of customers decreases from $k + 1$ to k, and moreover T_k is the first time at which this happens. Therefore, the last departure of a customer before T_k must occur no earlier than the last arrival before T_k. So b_k, being the service time of the customer who departs at T_k must be no more than the time between T_k and the last arrival before T_k. The latter time period is $T_k - a[T_k/a]$. The desired inequality now follows easily. ■

REFERENCE

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