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CONVERGENCE OF THE ISHIKAWA-TYPE ITERATION PROCEDURE FOR MULTI-VALUED OPERATORS OF THE ACCRETIVE TYPE*

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We prove that our recent result [*Soochow J. Math.* **22** (4), (1996), 485-94] dealing with the iterative construction of fixed points of certain multi-valued operators of the accretive type by Mann-type iteration procedure can be extended to the Ishikawa-type iteration procedure. In our present result, our iteration parameters are completely independent of any geometric properties of the underlying Banach spaces.

Key Words : Multi-valued Operators; Accretive Operators; ψ-Strong Pseudocontractions; Uniformly Smooth Spaces; Fixed Points; Mann Iteration; Ishikawa Iteration

1. INTRODUTION

Let E be a Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $Jx = \{f^* \in E^* : \|f^*\|^2 = \|x\|^2 = \langle x, f^* \rangle\}$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If E^* is uniformly convex, then J is single-valued and is uniformly continuous on any bounded subset of E. In the sequel we shall denote single-valued normalized duality map by j.

Let K be a nonempty subset of E. Recently, the author[1] studied the class of multivalued operators $T : K \to 2^K$ satisfying the condition

$$Re \langle \xi - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2 - \phi(\|x - x^*\|) \qquad \dots (1)$$

for all $x \in K$, $\xi \in Tx$ and for some $x^* \in K$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$. It is shown in [1] that the class of mappings studied by Dunn[2], Chidume[3] and Weng[4] is a proper subclass of the class of mappings satisfying (1). We shall also show in this paper that the class of mappings satisfying (1) include the important class of multi-valued ψ-strong pseudocontractions with nonempty fixed-point sets.

It is well known (see for example[5]) that if E is a uniformly smooth Banach space, then there exists a continuous nondecreasing function $b : [0, \infty) \to [0, \infty)$ such that $b(0) = 0$, $b(ct) \leq cb(t)$ for all $c \geq 1$, and

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$$\| x+y \|^2 \leq \| x \|^2 + 2\, Re \langle y, j(x) \rangle + \max \{\| x \|, 1\} \; \| y \| \, b(\| y \|) \qquad \ldots (2)$$

for all $x, y \in E$.

In [1], the author proved the following :

Theorem M ([1], p. 489) — *Suppose K is a nonempty subset of a uniformly smooth Banach space and suppose $T : K \to 2^K$ satisfies (1), i.e;*

$$Re \langle \xi - x^*, j(x-x^*) \rangle \leq \| x - x^* \|^2 - \phi(\| x - x^* \|),$$

for all $x \in K$, $\xi \in Tx$ and for some $x^* \in K$ and a strictly increasing function $\phi : [o, \infty) \to [0, \infty)$ with $\phi(0) = 0$. If y^* is a fixed point of T then $y^* = x^*$, thus T can have at most one fixed point. Suppose the range of T is bounded and suppose $\{x_n\}_{n=0}^{\infty} \subseteq K$ satisfies

$$x_{n+1} = (1 - c_n)x_n + c_n \xi_n, \quad \exists\, \xi_n \in Tx_n, \qquad \ldots (3)$$

with $\{\alpha_n\}_{n=0}^{\infty} \subseteq [0, 1]$ *satisfying*

$$(i) \lim_{n \to \infty} \alpha_n = 0,$$

$$(ii) \sum_{n=0}^{\infty} \alpha_n = \infty$$

and $$(iii) \sum_{n=0}^{\infty} \alpha_n b(\alpha_n) < \infty.$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^*.

Theorem M shows that the Mann-type iteration procedure (3) with the iteration parameter $\{\alpha_n\} \subseteq [0, 1]$ satisfying conditions (*i*)-(*iii*) of Theorem M converges strongly to x^* regardless of whether x^* is a fixed point of T or not.

The Mann-type iteration method and the more general method of the Ishikawa-type iteration procedure given by

$$y_n = (1 - \beta_n)\, x_n + \beta_n \eta_n, \exists\, \eta_n \in Tx_n, n \geq 0 \qquad \ldots (4)$$

and $$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \xi_n, \exists\, \xi_n \in Ty_n, n \geq 0, \qquad \ldots (5)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are suitable sequences in [0, 1] have been studied extensively by several authors (see for example [1-4], [6-18]). Observe that if $\beta_n = 0 \;\forall\; n \geq 0$ in (4) and (5) then the

Ishikawa iteration procedure reduces to the Mann iteration procedure. However, it is now well known that even though the two iteration procedures are similar, they may exhibit different behaviours for different classes of nonlinear mappings (see for example [18] for the detailed comparison of the two processes). It has, therefore, become increasingly interesting to investigate the behaviour of the two iteration procedures for any given class of nonlinear mappings.

While Theorem M generalizes Theorem 1 of Chidume[3] (which is itself a generalization of a theorem of Dunn[2], and extends Theorem 1 of Weng[4] (see Remark 3 of [1]), it is not known whether Theorem M is extendible to the more general Ishikawa iteration procedure.

Furthermore, condition *(iii)* of Theorem M is dependent on the geometry of the underlying Banach space and is therefore not convenient to check in applications in some uniformly smooth Banach spaces. Nevanlinna and Reich[19] have shown how to choose the iteration parameter $\{\alpha_n\}$ to satisfy condition *(iii)* when $E = L_p$ (or l_p), $1 < p < \infty$ (see for example Remark 1 and Corollaries 1 and 2 of [1]). However, condition *(iii)* is certainly inconvenient to verify in many uniformly smooth Banach spaces.

It is our purpose in this paper to extend Theorem M to the more general Ishikawa iteration procedure. Moreover, our iteration parameters $\{\alpha_n\}$ and $\{\beta_n\}$ will be completely independent of any geometric properties of the underlying Banach spaces and hence can easily be chosen at the start of the iteration process. Theorem M, Theorem 1 of Dunn[2], Theorem 1 of Chidume[3] and Theorem 1 of Weng[4] will be special cases of our Theorem.

We shall need the following :

Lemma HY ([13], p.1706) — Let E be a Banach space with a strictly convex dual. Then for all $x, y \in E$ we have

$$\| x + y \|^2 \leq \| x \|^2 + 2 \operatorname{Re} \langle y, j(x - y) \rangle. \qquad \dots (6)$$

2. Main Results

Theorem — *Let E be a uniformly smooth Banach space and let K be a nonempty subset of E. Let $K \to 2^K$ satisfy (1), i.e.,*

$$Re \ \langle \xi - x^*, j(x - x^*) \rangle \leq \| x - x^* \|^2 - \phi (\| x - x^* \|),$$

for all $x \in K$, $\xi \in Tx$ and for some $x^* \in K$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$. If y^* is a fixed point of T, then $y^* = x^*$, thus T can have at most one fixed point. Suppose the range of T is bounded and let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions:

(i) $0 \leq \alpha_n, \beta_n \leq 1, n \geq 0,$

(ii) $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0$ *and*

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty.$

Let $\{x_n\}_{n=0}^{\infty} \subseteq K$ *satisfy*

$$y_n = (1-\beta_n)x_n + \beta_n\eta_n,\ \exists\, \eta_n \in Tx_n,\ n \ge 0$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\xi_n,\ \exists\, \xi_n \in Ty_n,\ n \ge 0.$$

Then $\{x_n\}$ *converges strongly to* x^*.

PROOF : It follows as in the proof of the Theorem of [1] that if y^* is a fixed point of T then $y^* = x^*$.

Using (6) we obtain

$$\begin{aligned}
\| x_{n+1} - x^* \|^2 &= \| (1-\alpha_n)(x_n - x^*) + \alpha_n(\xi_n - x^*) \|^2 \\
&\le (1-\alpha_n)^2 \| x_n - x^* \|^2 + 2\alpha_n \operatorname{Re} \langle \xi_n - x^*, j(x_{n+1} - x^*) \rangle \\
&= (1-\alpha_n)^2 \| x_n - x^* \|^2 + 2\alpha_n \operatorname{Re} \langle \xi_n - x^*, j(y_n - x^*) \rangle \\
&\qquad + 2\alpha_n \operatorname{Re} \langle \xi_n - x^*, j(x_{n+1} - x^*) - j(y_n - x^*) \rangle \\
&\le (1-\alpha_n)^2 \| x_n - x^* \|^2 + 2\alpha_n \| y_n - x^* \|^2 - 2\alpha_n \phi(\| y_n - x^* \|) \\
&\qquad + 2\alpha_n \operatorname{Re} \langle \xi_n - x^*, j(x_{n+1} - x^*) - j(y_n - x^*) \rangle \\
&\le (1-\alpha_n)^2 \| x_n - x^* \|^2 + 2\alpha_n \| y_n - x^* \|^2 - 2\alpha_n \phi(\| y_n - x^* \|) \\
&\qquad + 2\alpha_n \| \xi_n - x^* \| \| j(x_{n+1} - x^*) - j(y_n - x^*) \|. \quad \dots (7)
\end{aligned}$$

Observe that

$$\begin{aligned}
\| y_n - x^* \|^2 &= \| (1-\beta_n)(x_n - x^*) + \beta_n(\eta_n - x^*) \|^2 \\
&\le (1-\beta_n)^2 \| x_n - x^* \|^2 + 2\beta_n \operatorname{Re} \langle \eta_n - x^*, j(y_n - x^*) \rangle \\
&\le (1-\beta_n)^2 \| x_n - x^* \|^2 + 2\beta_n \| \eta_n - x^* \| \| y_n - x^* \|. \quad \dots (8)
\end{aligned}$$

Using (8) in (7) we obtain

$$\begin{aligned}
\| x_{n+1} - x^* \|^2 &\le [(1-\alpha_n)^2 + 2\alpha_n(1-\beta_n)^2] \| x_n - x^* \|^2 - 2\alpha_n \phi(\| y_n - x^* \|) \\
&\qquad + 4\alpha_n\beta_n \| \eta_n - x^* \| \| y_n - x^* \| + 2\alpha_n \| \xi_n - x^* \| \| j(x_{n+1} - x^*) - j(y_n - x^*) \|
\end{aligned}$$

$$\le [1+\alpha_n^2] \| x_n - x^* \|^2 - 2\alpha_n \phi(\| y_n - x^* \|) + 4\alpha_n\beta_n \| \eta_n - x^* \| \| y_n - x^* \|$$

$$+ 2\alpha_n \| \xi_n - x^* \| \| j(x_{n+1} - x^*) - j(y_n - x^*) \|. \quad \dots (9)$$

Let $M := \sup \{\| \xi_n - x^* \| + \| \eta_n - x^* \| : n \ge 0\} + \| x_0 - x^* \|$. We prove by induction that

$\| x_n - x^* \| \le M, \ \forall\ n \ge 1$. For $n = 1$, we have

$\| x_1 - x^* \| = \| (1-\alpha_0)(x_0 - x^*) + \alpha_0 (\xi_0 - x^*) \| \le (1-\alpha_0) \| x_0 - x^* \| + \alpha_0 \|$.
$\| \xi_0 - x^* \| \le (1-\alpha_0) M + \alpha_0 M = M.$

Assume now that $\| x_k - x^* \| \le M$ for some integer $k > 1$. Then

$\| x_{k+1} - x^* \| \le (1-\alpha_k) \| x_k - x^* \| + \alpha_k \| \xi_k - x^* \| \le (1-\alpha_k) M + \alpha_k M = M$, so that $\| x_n - x^* \| \le M, \ \forall\ n \ge 1$.

Since it follows from the definition of M that $\| x_0 - x^* \| \le M$, then we have

$$\| x_n - x^* \| \le M, \ \forall\ n \ge 0. \quad \dots (10)$$

Hence

$$\| y_n - x^* \| \le (1-\beta_n) \| x_n - x^* \| + \beta_n \| \eta_n - x^* \| \le (1-\beta_n) M + \beta_n M = M, \ \forall\ n \ge 0 \quad \dots (11)$$

Using (10) and (11) in (9) we obtain

$$\| x_{n+1} - x^* \|^2 \le \| x_n - x^* \|^2 - 2\alpha_n \phi(\| y_n - x^* \|) + M^2\alpha_n^2 + 4M^2 \alpha_n\beta_n$$

$$+ 2M\alpha_n \| j(x_{n+1} - x^*) - j(y_n - x^*) \|$$

$$= \| x_n - x^* \|^2 - 2\alpha_n \phi(\| y_n - x^* \|) + \alpha_n \lambda_n, \quad \dots (12)$$

where $\lambda_n = M(\alpha_n M + 4M\beta_n + 2 \| j(x_{n+1} - x^*) - j(y_n - x^*) \|)$.

Observe that

$$\| (x_{n+1} - x^*) - (y_n - x^*) \| = \| x_{n+1} - y_n \|$$

$$= \| (1-\alpha_n)(x_n - y_n) + \alpha_n (\xi_n - y_n) \|$$

$$\le (1-\alpha_n) \beta_n \| \eta_n - x_n \| + \alpha_n [\| \xi_n - x^* \| + \| y_n - x^* \|]$$

$$\leq \beta_n [\| \eta_n - x^* \| + \| x_n - x^* \|] + \alpha_n [\| \xi_n - x^* \| + \| y_n - x^* \|]$$

$$\leq 2M (\beta_n + \alpha_n) \to 0 \text{ as } n \to \infty.$$

Since $\{x_{n+1} - x^*\}$ and $\{y_n - x^*\}$ are bounded and j is uniformly continuous on bounded subsets of E, we have

$$\lim_{n \to \infty} \| j(x_{n+1} - x^*) - j(y_n - x^*) \| = 0.$$

Hence $$\lim_{n \to \infty} \lambda_n = 0.$$

Let $\inf \{\| y_n - x^* \| : n \geq 0\} = \delta \geq 0$. We prove that $\delta = 0$. Assume, for contradiction that $\delta > 0$. Then $\| y_n - x^* \| \geq \delta > 0$, $\forall\, n \geq 0$. Hence,

$$\phi(\| y_n - x^* \|) \geq \phi(\delta) > 0, \ \forall\, n \geq 0.$$

Thus we obtain from (12) that

$$\| x_{n+1} - x^* \|^2 \leq \| x_n - x^* \|^2 - \alpha_n \phi(\delta) - \alpha_n [\phi(\delta) - \lambda_n] \ \forall\, n \geq 0.$$

Since $\lim_{n \to \infty} \lambda_n = 0$, there exists a positive integer N such that $\lambda_n \leq \phi(\delta)$ $\forall\, n \geq N$. Hence

$$\| x_{n+1} - x^* \|^2 \leq \| x_n - x^* \|^2 - \alpha_n \phi(\delta), \ \forall\, n \geq N,$$

so that

$$\alpha_n \phi(\delta) \leq \| x_n - x^* \|^2 - \| x_{n+1} - x^* \|^2, \ \forall\, n \geq N.$$

Hence,

$$\phi(\delta) \sum_{j=N}^{n} \alpha_j \leq \| x_N - x^* \|^2, \ \forall\, n \geq N,$$

so that $\sum_{n=0}^{\infty} \alpha_n < \infty$, contradicting condition (iii). Hence $\inf \{\| y_n - x^* \| : n \geq 0\} = \delta = 0$. Thus there exists a subsequence $\left\{ \| y_{n_j} - x^* \| \right\}_{j=0}^{\infty}$ of $\left\{ \| y_{n_j} - x^* \| \right\}_{n=0}^{\infty}$ such that $\lim_{j \to \infty} \| x_{n_j} - x^* \| = 0$. Since $\left\{ \eta_{n_j} - x_{n_j} \right\}_{j=0}^{\infty}$ is bounded and

$$x_{n_j} - x^* = y_{n_j} - x^* - \beta_{n_j} (\eta_{n_j} - x_{n_j}),$$

we have $\lim_{j\to\infty} \| x_{n_j} - x^* \| = 0$. Thus given any $\varepsilon > 0$, there exists a positive intger j_0 such that

$$\| x_{n_j} - x^* \| < \varepsilon, \ \forall\, j \geq j_0 \text{ (i.e. } \forall\, n_j \geq n_{j_0}).$$

Moreover, since $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \lambda_n = 0$, there exists a positive integer N_0 such that for all $n \geq N_0$ we have

$$\alpha_n \leq \frac{\varepsilon}{8M},\ \beta_n \leq \frac{\varepsilon}{8M} \text{ and } \lambda_n \leq 2\phi\left(\frac{\varepsilon}{2}\right).$$

Since $n_j \to \infty$ *as* $j \to \infty$, choose j_* sufficiently large such that $n_{j*} \geq \max\ \{n_{j0}, N_0\}$. Then for all $n \geq n_{j*}$ we have

$$\| x_{n_{j*}} - x^* \| < \varepsilon,\ \alpha_n \leq \frac{\varepsilon}{8M},\ \beta_n \leq \frac{\varepsilon}{8M} \text{ and } \lambda_n \leq 2\phi\left(\frac{\varepsilon}{2}\right).$$

Claim — $\| x_{n_{j_*}+k} - x^* \| < \varepsilon$ for all integers $k \geq 1$. The proof of the claim is by induction. For $k = 1$, we prove that $\| x_{n_{j_*}+1} - x^* \| < \varepsilon$. Assume, for contradiction, that

$\| x_{n_{j_*}+1} - x^* \| \geq \varepsilon$. Then,

$$\| x_{n_{j_*}} - x^* \| \geq \| x_{n_{j_*}+1} - x^* \| - \alpha_{n_{j_*}} \| x_{n_{j_*}} - \xi_{n_{j_*}} \|$$

$$\geq \varepsilon - 2M\,\alpha_{n_{j_*}} \geq \frac{3\varepsilon}{4}.$$

Furthermore,

$$\| y_{n_{j_*}} - x^* \| \geq \| x_{n_{j_*}} - x^* \| - \beta_{n_{j_*}} \| x_{n_{j_*}} - \eta_{n_{j_*}} \|$$

$$\geq \frac{3\varepsilon}{4} - 2M\,\beta_{n_{j_*}} \geq \frac{\varepsilon}{2}.$$

It follows from (12) that

$$\varepsilon^2 \leq \| x_{n_{j_*}+1} - x^* \|^2 \leq \| x_{n_{j_*}} - x^* \|^2 - 2\alpha_{n_{j_*}} \phi(\| y_{n_{j_*}} - x^* \|) + \alpha_{n_{j_*}} \lambda_{n_{j_*}}$$

$$\leq \| x_{n_{j_*}} - x^* \|^2 - 2\alpha_{n_{j_*}} \phi\left(\frac{\varepsilon}{2}\right) + 2\alpha_{n_{j_*}} \phi\left(\frac{\varepsilon}{2}\right)$$

$$= \| x_{n_{j_*}} - x^* \|^2 < \varepsilon^2,$$

a contradiction, so that $\| x_{n_{j_*}+1} - x^* \| < \varepsilon$.

Assume now that $\| x_{n_{j_*}+k_0} - x^* \| < \varepsilon$ for some $k_0 > 1$. We prove that $\| x_{n_{j_*}+(k_0+1)} - x^* \| < \varepsilon$. Assume, for contradiction, that $\| x_{n_{j_*}+(k_0+1)} - x^* \| \geq \varepsilon$. Then

$$\| x_{n_{j_*}+k_0} - x^* \| \geq \| x_{n_{j_*}+(k_0+1)} - x^* \| - \alpha_{n_{j_*}+k_0} \| x_{n_{j_*}+k_0} - \xi_{n_{j_*}+k_0} \|$$

$$\geq \varepsilon - 2M\, \alpha_{n_{j_*}+k_0} \geq \frac{3\varepsilon}{4}.$$

Furthermore,

$$\| y_{n_{j_*}+k_0} - x^* \| \geq \| x_{n_{j_*}+k_0} - x^* \| - \beta_{n_{j_*}+k_0} \| x_{n_{j_*}+k_0} - \eta_{n_{j_*}+k_0} \|$$

$$\geq \frac{3\varepsilon}{4} - 2M\, \beta_{n_{j_*}+k_0} \geq \frac{\varepsilon}{2}.$$

Using (12) we obtain

$$\varepsilon^2 \leq \| x_{n_{j_*}+(k_0+1)} - x^* \|^2$$

$$\leq \| x_{n_{j_*}+k_0} - x^* \|^2 - 2\alpha_{n_{j_*}+k_0} \phi(\| y_{n_{j_*}+k_0} - x^* \|) + \alpha_{n_{j_*}+k_0} \lambda_{n_{j_*}+k_0}$$

$$\leq \| x_{n_{j_*}+k_0} - x^* \|^2 - 2\alpha_{n_{j_*}+k_0} \phi\left(\frac{\varepsilon}{2}\right) + 2\alpha_{n_{j_*}+k_0} \phi\left(\frac{\varepsilon}{2}\right)$$

$$\leq \| x_{n_{j_*}+k_0} - x^* \|^2 < \varepsilon^2,$$

a contradiction, so that $\| x_{n_{j_*}+k} - x^* \| < \varepsilon$. Hence it follows by induction that

$$\| x_{n_{j_*}+k} - x^* \| < \varepsilon \text{ for all integers } k \geq 1,$$

and this implies that $\lim_{n \to \infty} \| x_n - x^* \| = 0$, completing the proof of the Theorem.

If we set $\beta_n = 0$, $\forall\, n \geq 0$ in our Theorem we obtain :

Corollary 1 — Let E be a uniformly smooth Banach space and let K be a nonempty subset of E. Let $T: K \to 2^K$ satisfy (1). If y^* is a fixed point of T, then $y^* = x^*$, thus T can have at most one fixed point. Suppose the range of T is bounded and let $\{\alpha_n\}$ be a real sequence satisfying the conditions :

(*i*) $0 \le \alpha_n \le 1, n \ge 0,$

(*ii*) $\lim_{n \to \infty} \alpha_n = 0$, and

(*iii*) $\sum_{n=0}^{\infty} \alpha_n = \infty.$

Let $\{x_n\} \subseteq K$ satisfy

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n \xi_n, \exists \in Tx_n, n \ge 0.$$

Then $\{x_n\}$ converges strongly to x^*.

Remark 1 : If K is convex and $Tx \ne \phi$ for all $x \in K$, it follows as in [2] that for every $x_0 \in K$ and real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in [0, 1], there is at least one sequence $\{x_n\}$ which begins at x_0 and satisfies

$$y_n = (1 - \beta_n)x_n + \beta_n \eta_n, \exists\ \eta_n \in Tx_n, n \ge 0$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \xi_n, \exists\ \xi_n \in Ty_n, n \ge 0.$$

Furthermore, if K is convex and the range of T is bounded, then there is at most one $x^* \in K$ satisfying (1) regardless of whether or not T has a fixed point.

Remark 2 : A mapping T with domain $D(T)$ in a Banach space E and range $R(T)$ in 2^E is called a strong pseudocontraction if for all $x, y \in D(T), \xi \in Tx, \eta \in Ty$, there exist $j(x-y) \in j(x-y)$ and a constant $t > 1$ such that

$$Re \langle \xi - \eta, j(x-y) \rangle \le \frac{1}{t} \| x - y \|^2.$$

T is called a ψ-strong pseudocontraction if for all $x, y \in D(T), \xi \in Tx, \eta \in Ty$, there exist a $j(x-y) \in J(x-y)$ and a strictly increasing function $\psi: [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that

$$Re \langle \xi - \eta, j(x-y) \rangle \le \| x - y \|^2 - \psi(\| x - y \|) \| x - y \|.$$

Furthermore, T is called a ψ-hemicontraction if $F(T) = \{x \in D(T) : x \in Tx\} \ne 0$ and for all $x \in D(T), x^* \in F(T), \xi \in Tx, \eta^* \in Tx^*$, there exist $j(x - x^*) \in J(x - x^*)$ and a strictly increasing function $\psi: [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that

$$Re \langle \xi - \eta, j(x - x^*) \rangle \leq \| x - x^* \|^2 - \psi(\| x - x^* \|) \| x - x^* \|. \quad \ldots (13)$$

It is shown in [11] that the class of strong pseudocontractions is a proper subclass of the class of ψ-strong pseudocontractions. An example in [8] shows that the class of ψ-strong pseudocontractions with a nonempty fixed-point set $F(T)$ is a proper subclass of the class of ψ-hemicontractions.

Since $x^* \in Tx^*$, if we set $\eta^* = x^*$ in (13) we obtain

$$Re \langle \xi - x^*, j(x - x^*) \rangle \leq \| x - x^* \|^2 - \psi(\| x - x^* \|) \| x - x^* \| = \| x - x^* \|^2 - \phi(\| x - x^* \|) \quad ,$$

where $\phi(\| x - x^* \|) := \psi(\| x - x^* \|) \| x - x^* \|$. Since $\phi : [0, \infty) \rightarrow [0, \infty)$ defined by $\phi(t) := \psi(t)t$ is strictly increasing and $\phi(0) = 0$ we have that every ψ-hemicontraction satisfies (1). Thus we have the following :

Corollary 2 — Let E be a uniformly smooth Banach space, and let K be a nonempty subset of E. Let $T : K \rightarrow 2^K$ be a ψ-hemicontraction. Then $F(T)$ is singleton. Suppose the range of T is bounded and let $\{\alpha_n\}$ and $\{\beta_n\}$ be as in our Theorem. Let $\{x_n\}$ satisfy

$$y_n = (1 - \beta_n)x_n + \beta_n \eta_n, \ \eta_n \in Tx_n, \ n \geq 0$$

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n \xi_n, \ \xi_n \in Ty_n, \ n \geq 0.$$

Then the sequence $\{x_n\}$ converges strongly to the fixed point of T.

Remark 3 : Our Theorem and the Corollaries extend and generalize several results that have appeared recently. In particular, the results of ([1], [3]), Theorem 1 of Dunn [2], Theorem 1 of Chidume [3], Theorem 1 of Weng [4] and a host of other results are special cases of our Theorem. Moreover, our iteration parameters $\{\alpha_n\}$ and $\{\beta_n\}$ are completely independent of any geometric properties of the underlying Banach spaces and hence can easily be chosen at the start of the iteration process. Prototype for our iteration parameters is $\alpha_n = \beta_n = \frac{1}{n+1} \ \forall \ n \geq 0$.

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