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CORE EQUALITY THEOREMS FOR REAL BOUNDED SEQUENCES.

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In this paper, we have studied the matrix classes leaving the core of a bounded sequence invariant.

Key Words : Core Equality; Bounded sequence; Infinite Matrices, Almost Convergences

1. INTRODUCTION

Let m, c, c_0 be the linear spaces of real bounded, convergent and null sequences $x = \{x_n\}$, respectively, normed by $\| x \| = \sup | x_n |$. We write

$$m_0 = \left\{ x \in m : \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}.$$

We define the functionals l and L on m by

$$l(x) = \liminf x_n \;;\; L(x) = \limsup x_n.$$

Let $A = (a_{nk})$ be an infinite matrix and we write

$$(Ax)_n := \sum_k a_{nk} x_k$$

if the series converges for each $n \geq 0$. By Ax we denote the sequence $\{(Ax)_n\}$. If $\lim Ax = \lim x$ for each $x \in c$, we say that A is regular[2, 11] and write $A \in (c, c; p)$. Silverman Toeplitz theorem gives the necessary and sufficient conditions for regularity of the matrix A[2, 11].

A matrix $A = (a_{nk})$ is called normal if it is lower semi triangular matrix with non-zero diagonal entries[2].

A matrix $A = (a_{nk})$ is called strongly regular[8] if it is regular and

$$\lim_n \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| = 0$$

A regular matrix A is almost positive[3] if and only if

$$\lim_n \sum_{k=0}^{\infty} |a_{nk}| = 1.$$

The famous Knopp's theorem (see, e.g. [2], [4], [6], [9], [12]) determines a class of regular matrices for which $L(Ax) \leq L(x)$ for all $x \in m$; that is κ-core $\{Ax\} \subseteq \kappa$ - core $\{x\}$ where κ - core $\{x\}$, denotes Knopp's core of $x \in m$ defined by the closed interval $[l(x), L(x)]$.

It is well known that the functional

$$q(x) = \inf_{n_1, n_2, \ldots, n_r} \limsup_k \frac{1}{r} \sum_{i=1}^{r} x_{k+n_i}$$

is sublinear on m[8,11].

Following [10], we consider the functionals

$$l^*(x) = \liminf_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r, \; L^*(x) = \limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r$$

on m.

If $q(x) = q(-x) = s$, then x is called almost convergent[8] to s and in this case we write $F - \lim x = s$. F and F_o respectively denote the set of all almost convergent and all almost convergent to zero sequences. It is also well known that $q(x) = L^*(x)$[5].

Das[4] defined the Banach core of $x \in m$ to be the closed interval $[-q(-x), q(x)]$ (see also [7]). By β - *core* $\{x\}$ we denote the Banach core of $x \in m$

. If $F - \lim Ax = F - \lim x$ for each $x \in m$, we say that A is F – regular[6].

Orhan[10] established a clss of F – regular matrices for which $L^*(Ax) \leq L^*(x)$ i.e., β – core $\{Ax\} \subseteq \beta$ – core $\{x\}$ for all $x \in m$.

Allen[1] determined the class of matrices under which the κ - core of every bounded sequence is identical with the κ - core of its transform.

In the present paper we study the core equality problems.

2. The Main Results

In this section we assume that all matrices have finite norm.

Theorem 1 — *Let* $A = (a_{nk})$ *be a real matrix and* $x = (x_n)$ *real bounded sequence. Then*

κ - *core* $\{Ax\} = \beta$ – *core* $\{x\}$

if and only if

a) A is strongly regular

b) $\sum_k |a_{nk}| \to 1 \ (n \to \infty)$

and *c) for every infinite sequence of suffixes* p_i *(i = 1, 2,...), the number 1 is a limit point of the sequence* $(u_n) = (\sum_i a_{n,p_i})$.

PROOF : Necessity; Let κ - *core* $\{Ax\} \subset \beta$ - *core* $\{x\}$. Hence, $LA \leq L^*$. From [10], Theorem 3 (a) and (b) are satisfied.

Now let (p_i) be any given sequence of suffixes. We define a bounded sequence $x = (x_k)$ by $x_{p_i} = 1$ for every i and $x_k = 0$ otherwise. If $\{x_k\}$ has a finite number of zero terms then lim x = 1 and so, F − lim x = 1. In this case β−*core* $\{x\}$ = $\{1\}$ and from κ - *core* $\{Ax\} \subset \beta$−*core* $\{x\}$ we have κ - *core* $\{Ax\} \subseteq \{1\}$. Since Ax is bounded we must have lim Ax = 1.

If $\{x_k\}$ has an infinite number of zero terms, then κ - *core* $\{x\}$ = [0, 1]. In this case, there exists subsequences $\{u_j\}$ and $\{y_v\}$ such that

$$u_j := x_{k_j} \to 1 \ (j \to \infty)$$

$$y_v := x_{k_v} \to 0 \ (v \to \infty)$$

also 1 and 0 are respectively F - limit points of subsequences (u_j) and (y_v). Hence,

$$[0, 1] \subset \beta - core \ \{x\} \subset \kappa - core \ \{x\} = [0, 1].$$

By assumption we have

$$\kappa \text{ - } core \ \{Ax\} = \beta - core \ \{x\} = [0, 1].$$

It follows that (c) is necessary.

Sufficiency : Let (a), (b) and (c) hold. Then it follows Theorem 3 of Orhan[10] that $LA \leq L^*$, i.e. κ - *core* $\{Ax\} \subset \beta$−*core* $\{x\}$ for every $x \in m$. Now we shall prove that β − *core* $\{x\}$ $\subset \kappa$ − *core* $\{Ax\}$. In order to do so, we must show that the set of all F-limit points of the sequence $\{x_k\}$ is a subset of the set of limit points of $\{(Ax)_n\}$.

Let $\{x_k\}$ be bounded and let α be an F-limit point of x. Then there exist a sequence $\{p_i\}$ of suffixes such that $F - \lim_i x_{p_i} = \alpha$. Given any $\varepsilon > 0$ we put $x_{pi} = \alpha + \varepsilon_{pi}$ such that $F - \lim \varepsilon_{pi} = 0$.

Let $q_j \ (j = 1, 2, \ldots)$ be the sequence obtained by, omitting the italic number p_i from the sequence (1, 2, 3, ...). Now we can write

$$y_n = \sum_i a_{n,p_i} x_{p_i} + \sum_j a_{n,q_j} x_{q_j}.$$

we see from (c) that there is a sequence $\{m_r\}$ of positive integers such that

$$\sum_i a_{m_r} = u_{m_r} \to 1$$

when $r \to \infty$ and we have

$$y_{m_r} = \sum_i a_{m_r, p_i} + \sum_j a_{m_r, q_j} x_{q_j}.$$

And also we have

$$y_{m_r} - \alpha = \sum_{i=1}^{N} a_{m_r, p_i} x_{p_i} + \alpha \left(\sum_{i=N+1}^{\infty} a_{m_r, p_i} - 1 \right) + \sum_{i=N+1}^{\infty} a_{m_r, p_i} \varepsilon_{p_i} + \sum_j a_{m_r, q_j} x_{q_j}.$$

We shall prove that $\lim_{r \to \infty} y_{m_r} = \alpha$. Now we have

$$|y_{m_r} - \alpha| \leq \|x\|_\infty \sum_{i=1}^{N} |a_{m_r p_i}| + |\alpha| \left| \sum_{i=N+1}^{\infty} a_{m_r, p_i} - 1 \right| + \left| \sum_{i=N+1}^{\infty} a_{m_r, p_i} \varepsilon_{p_i} \right| + \|x\|_\infty \sum_j |a_{m_r, q_j}|$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4$$

By regularity of A, we get $\sum_1 \to 0, (r \to \infty)$.

From (c) we have $\sum_{i=N+1}^{\infty} a_{m_r, p_i} \to 1 \ (r \to \infty)$ hence $\sum_2 \to 0 \ (r \to \infty)$.

We know that $F - \lim \varepsilon_{pi} = 0$. Let $A' := (a_{m_r, p_i})$. Since $A \in (F, c; p)$, it is easy to see that $A' \in (F_o, c_o)$. Hence, $\sum_3 \to 0, (r \to \infty)$.

Finally, we shall show that $\sum_4 \to 0$.

From (b) we have

$$\sum_k |a_{m_{r,k}}| \to 1.$$

Hence,

$$\sum_k |a_{m_{r,k}}| \le 1+\varepsilon$$

for all $r > r_1(\varepsilon)$. Since

$$\sum_k |a_{m_r}| = \sum_i |a_{m_r, p_i}| + \sum_j |a_{m_r, q_j}|. \quad \ldots (1)$$

It follows that

$$\sum_i |a_{m_r, p_i}| \le 1+\varepsilon \quad \ldots (2)$$

for all $r > r_2(\varepsilon)$.

From (c) we get

$$\sum_i a_{m_r, p_i} = u_{m_r} \to 1$$

hence we have $|u_{m_r} - 1 \le \varepsilon|$, so $\| u_{m_r} | - | 1 || \le \varepsilon$, hence

$$1-\varepsilon \le |u_{m_r}| = \left| \sum_i a_{m_r, p_i} \right| \le \sum_i \left| a_{m_r, p_i} \right|$$

for all $e > r_2(\varepsilon)$.

From (2) and the last inequality, for $r > r_o(\varepsilon) :=$ max $\{r_1, r_2\}$ we have

$$1-\varepsilon \le \sum_i |a_{m_r, p_i}| \le 1+\varepsilon$$

and so

$$\sum_i |a_{m_r, p_i}| \to 1.$$

Hence, from (1) we have

$$\sum_j \left| a_{m_r, q_j} \right| \to 0.$$

In this case $|y_{m_r} - \alpha| < \varepsilon$ and so α is a limit point of Ax. It follows that

$$\beta - core\ \{x\} \subset \kappa - core\ \{Ax\}$$

Hence we have

$$\beta - core\ \{x\} \subset \kappa - core\ \{Ax\}$$

which proves the theorem.

We write

$$m_B := \{x : Bx \in m\}.$$

Before giving the equality $L^*(Ax) = L^*(Bx)$, we need the following result[13]:

Theorem A — *Let* B *be a normal matrix. Then for a row-finite matrix*

$$L^*(Ax) \leq L^*(Bx) \quad (\text{for all } x \in m_B)$$

if and only if

a) $C = AB^{-1}$ *is F-regular (i.e., F – lim Cx = F – lim x for each* $x \in F$*)*

$$b) \limsup_{n} \sup_{i} \sum_{k} \left| \frac{1}{n+1} \sum_{r=i}^{i+n} c_{rk} \right| = 1.$$

If we interchange the roles of the matrices A and B, we immediately get the following :

Theorem 2 — *Let* $B = (b_{nk})$ *and* $A = (a_{nk})$ *be a normal matrices. Then we have for all* $x \in m_B \cap m_A$ *that* $L^*(Ax) = L^*(Bx)$ *if and only if* $C = AB^{-1}$ *and* $D = BA^{-1}$ *are F-regular and*

$$\limsup_{n} \sup_{i} \sum_{k} \left| \frac{1}{n+1} \sum_{r=i}^{i+n} c_{rk} \right| = 1$$

and

$$\limsup_{n} \sup_{i} \sum_{k} \left| \frac{1}{n+1} \sum_{r=i}^{i+n} d_{rk} \right| = 1.$$

Before stating our final result we need the following result of Choudhary[3].

Theorem B — *Let B be a normal matrix. Then for a row finite matrix A,*

$$L(Ax) \leq L(Bx)$$

for all $x \in m_B$ *if and only if* AB^{-1} *is a regular and almost positive*[3].

Interchanging the roles of A and B we get the following ;

Theorem 3 — *Let* $B = (b_{nk})$ *and* $A = (a_{nk})$ *be a normal matrices. Then we have* $L(Ax) = L(Bx)$ *for all* $x \in m_B \cap m_A$ *if and only if the matrices* $C = AB^{-1}$ *and* $D = BA^{-1}$ *are regular and almost positive.*

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