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THE EXISTENCE OF SOLUTIONS FOR FUZZY DIFFERENTIAL EQUATIONS WITH INFINITE DELAYS*

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The existence of solutions, the existence of T-periodic solutions and almost periodic solutions for the fuzzy functional differential equation with infinite delays are proved.

Key Words : Fuzzy Functional Differential Equations; Infinite Time Delays; Fuzzy Mapping; Existence of Solutions; Uniformly Stable; *T*-periodic Solution; Almost Periodic Solution

1. INTRODUCTION

Seikkala[9] defined the fuzzy derivative which is the generalization of the Hukuhara derivative in [7], the fuzzy integral which is the same as Dubois and Prade[2], and by means of the extension principle of Zadeh, showed that the fuzzy initial value problem $x'(t) = f(t, x(t)), x(0) = x_0$ has a unique fuzzy solution when f satisfies the generalized Lipschitz condition which guarantees a unique solution of the deterministic initial value problem. Kaleva[3] studied the Cauchy problem of fuzzy differential eqations, characterized those subsets of fuzzy sets in which the peano theorem is valid. Park, Kwun and Jeong[6] consider the existence of solutions of fuzzy integral equations in Banach space and Subramanyam and Sudarsanam[10] proved the existence of solutions fuzzy equations functional.

By $C((-\infty, 0]; E^n)$ denotes the set of all fuzzy continuous mappings from $(-\infty, 0]$ to E^n with $D(\phi(t), \hat{0}) \leq M$ for some constant M and for $\hat{0} \in E^n$. Here we denote

$$D(\phi, \hat{0}) = \sup \{D(\phi(t), \hat{0}) : t \in (-\infty, 0]\} \text{ for } \phi \in C((-\infty, 0]; E^n), \hat{0} \in E^n.$$

We write $\Gamma = C((-\infty, 0]; E^n)$ and for $\phi \in \Gamma$, set $D(\phi, \hat{0}) = D(\phi_{(-\infty, 0]}, \hat{0})$ for $\hat{0} \in E^n$. If $x(t)$ is a fuzzy mapping on $(-\infty, b), b \leq \infty$, we define for each $t \in (-\infty, b), x_t(s) = x(t+s), s \leq 0$. Clearly, if

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$x(t)$ is fuzzy continuous and bounded mapping on each interval $(-\infty, b_1], b_1 < b$, then $x_t \in \Gamma$ for $t \in (-\infty, b)$.

In this paper we consider the fuzzy functional differential equations with infinite time delays

$$x'(t) = F(t, x(t), x_t), \qquad \dots (1.1)$$

where F is a E^n-valued fuzzy mapping defined on $R \times E^n \times \Gamma$. The existence and stability of almost periodic solutions for functional differential equation of form (1.1) have been obtained by W. Quanyi[11]. The purpose of this paper is the existence of solutions, the existence of T-periodic solutions and almost periodic solutions for the fuzzy functional differential equations with infinite time delays.

2. Preliminaries

Let $P_K(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_K(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d(A, B) = \max\left\{ \sup_{a \in A} \inf_{B \in B} \| a - b \|, \ \sup_{b \in B} \inf_{a \in A} \| a - b \| \right\},$$

where $\| \cdot \|$ denote the usual Euclidean norm in R^n. Then it is clear that $(P_K(R^n), d)$ becomes a metric space.

Proposition 2.1[8] — The metric space $(P_K(R^n), d)$ is complete and separable.

We denote

$$E^n = \{u : R^n \to [0, 1] \mid u \text{ satisfies (i) - (iv) below}\},$$

where

(i) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous

and (iv) $[u]^0 = cl\ \{x \in R^n \mid u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$ denote $[u]^\alpha = \{x \in R^n \mid u(x) \geq \alpha\}$. Then from (i)-(iv), it follows that the α-level set $[u]^\alpha \in P_K(R^n)$ for all $0 \leq \alpha \leq 1$.

If $g : R^n \times R^n \to R^n$ is a function, then according to Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the equation

$$g(u, v)(z) = \sup_{z = g(x, y)} \min \{u(x), v(y)\}.$$

It is well known that

$$[g(u, v)]^{\alpha} = g([u]^{\alpha}, [v]^{\alpha})$$

for all $u, v \in E^n$, $0 \leq \alpha \leq 1$ and continuous function g. Especially for addition and scalar multiplication, we have

$$[u + v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}, \ [ku]^{\alpha} = k[u]^{\alpha},$$

where $u, v \in E^n, k \in R, 0 \leq \alpha \leq 1$.

Proposition 2.2[5] — If $u \in E^n$, then

(i) $[u]^{\alpha} \in P_K(R^n)$ for $0 \leq \alpha \leq 1$,

(ii) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$

and (iii) If $\{\alpha_k\} \subset [0, 1]$ is a non decreasing sequence converging to $\alpha > 0$, then

$$[u]^{\alpha} = \bigcap_{k \geq 1} [u]^{\alpha_k}.$$

Conversely, if $\{A^{\alpha} \mid 0 \leq \alpha \leq 1\}$ is a family of subsets of R^n satisfying (i)-(iii), then there exists a $u \in E^n$ such that

$$[u]^{\alpha} = A^{\alpha} \text{ for } 0 < \alpha \leq 1$$

and

$$[u]^0 = \overline{\bigcup_{0 < \alpha \leq 1} A^{\alpha}} \subset A^0.$$

Define $D : E^n \times E^n \to R^+ \cup \{0\}$ by the equations

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^{\alpha}, [v]^{\alpha}),$$

where d is the Hausdorff metric defined in $P_K(R^n)$. Then it is easy to show that D is a metric in E^n.

Using the result in S. Seikkala[9], we know that

(1) (E^n, D) is a complete metric space,

(2) $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$ and

(3) $D(ku, kv) = |k| D(u, v)$ for all $u, v \in E^n, k \in R$.

Definition 2.1 — Let $T=[t_0, t_0+p]\subset R$ be a compact interval. We say that a mapping $F: T\to E^n$ is strongly measurable if for all $\alpha\in[0,1]$ the set-valued mapping $F_\alpha : T\to P_K(R^n)$ defined by

$$F_\alpha(t)=[F(t)]^\alpha$$

is (Lebesgue) measurable, where $P_K(R^n)$ is endowed with the topology generated by the Hausdorff metric d. A mapping $F: T\to E^n$ is called integrably bounded if there exists an integrable function h such that $\|x\|\le h(t)$ for all $x\in F_0(t)$.

Definition 2.2 — Let $F: T\to E^n$. The integral of F over T denoted by $\int_T F(t)dt$ is defined level-wise by the equation

$$\left[\int_T F(t)dt\right]^\alpha=\int_T F_\alpha(t)dt$$

$$=\left\{\int_T f(t)dt \mid f: T\to R^n \text{ is a measurable selection for } F_\alpha\right\}$$

for all $0<\alpha\le 1$.

Proposition 2.3 — [1] If $F: T\to E^n$ is strongly measurable and integrably bounded, then F is integrable.

The following definitions and propositions are given in reference [4].

Proposition 2.4 — Let $F, G: T\to E^n$ be integrable and $\lambda\in R$. Then

(1) $\int_T (F(t)+G(t))dt=\int_T F(t)dt+\int_T G(t)dt;$

(2) $\int_T \lambda F(t)dt=\lambda\int_T F(t)dt,$

(3) $D(F, G)$ is integrable and

(4) $D\left(\int_T F(t)dt, \int_T G(t)dt\right)\le\int_T D(F, G)(t)dt.$

Remark 2.1 : Suppose $A\in E^n$ and define $F:[a,b]\to E^n$ by $F(s) = A$ for all $a\le s\le b$. Then from Example 4.1, we have

$$\int_a^b F(t)dt=(b-a)A.$$

Let $x, y \in E^n$. If there exists a $z \in E^n$ such that $x = y + z$, then we call z the H-difference of x and y, denoted by $x - y$

Definition 2.3 — A mapping $F : T \to E^n$ is differentiable at $t_0 \in T$, if there exists a $F'(t_0) \in E^n$ such that the limits

$$\lim_{h \to 0+} \frac{F(t_0 + h) - F(t_0)}{h} \text{ and } \lim_{h \to 0-} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist and are equal to $F'(t_0)$. Here the limit is taken in the metric space (E^n, D). At the end point of T, we consider only the one-side derivatives.

Proposition 2.5 — If $F, G : T \to E^n$ are differentiable and $\lambda \in R$, then

$$(F + G)'(t) = F'(t) + G'(t), (\lambda F)'(t) = \lambda F'(t).$$

Definition 2.4 — A mapping $F : T \to E^n$ is called continuous at $t_0 \in T$ is it is continuous at $t = t_0$ in the metric space (E^n, D).

Proposition 2.6 — Let $F : T \to E^n$ be continuous, then for every $t \in T$, the integral $G(t) = \int_{t_0}^{t} F(s)ds$ is differentiable and $G'(t) = F(t)$.

Proposition 2.7 — Let $F : T \to E^n$ be differentiable and assume that the derivative $F'(t)$ is integrable over T. Then for each $s \in T$ we have

$$F(s) = F(t_0) + \int_{t_0}^{s} F'(t)dt.$$

Proposition 2.8 — A mapping $F : T \to E^n$ is said to be uniformly continuous on T if it is uniformly continuous in the metric space (E^n, D).

Proposition 2.9 — A mapping $F : T \to E^n$ is said to be bounded if there is a real number M such that $D(f(t), \hat{0}) \leq M$ for all $t \in T, \hat{0} \in E^n$.

3. MAIN THEOREM

By $C((-\infty, 0]; E^n)$ denotes the set of all fuzzy continuous and bounded mappings from $(-\infty, 0]$ to E^n. Here we denote

$$D(\phi, \hat{0}) = \sup \{D(\phi(t), \hat{0}) : t \in (-\infty, 0]\} \text{ for } \phi \in C((-\infty, 0]; E^n), \hat{0} \in E^n.$$

We write $\Gamma = C((-\infty, 0]; E^n)$ and for $\phi \in \Gamma$, set $D(\phi, \hat{0}) = D(\phi_{(-\infty, 0]}, \hat{0})$ for $\hat{0} \in E^n$. If $x(t)$ is a fuzzy mapping on $(-\infty, b), b \leq \infty$, we define for each $t \in (-\infty, b), x_t(s) = x(t + s), s \leq 0$. Clearly, if

$x(t)$ is fuzzy continuous and bounded mapping on each interval $(-\infty, b_1], b_1 < b$, then $x_t \in \Gamma$ for $t \in (-\infty, b)$. The E^n-valued fuzzy mapping $F(t, x, \phi)$ on $R \times E^n \times \Gamma$ is said to satisfy following conditions:

(A_1) It is almost periodic fuzzy mapping in t uniformly for (x, ϕ) in closed bounded subsets of $E^n \times \Gamma$.

(A_2) There exists a $M > 0$ such that $D(F(t, 0, 0), \hat{0}) \le M$ for $t \in R, \hat{0} \in E^n$.

(A_3) The fuzzy mapping $F(t, x(t), x_t)$ is uniformly continuous on R whenever $x(t)$ is uniformly continuous with $D(x(t), \hat{0}) \le M'$ for some constant M', $\hat{0} \in E^n$.

(A_4) There exist positive numbers p, h and r such that $ph < 1, p \ge M/r$ where M is a constant in (A_2),

$$D(x(t) + hF(t, x(t), x_t), y(t) + hF(t, y(t), y_t)) \le (1 - ph)\, D(x_t, y_t) \qquad \text{... (3.1)}$$

for $t \in R$ and for any fuzzy mappings $x(t)$, $y(t)$ are uniformly continuous mappings on R with $D(x(t), \hat{0}) \le r, D(y(t), \hat{0}) \le r$ for $\hat{0} \in E^n$.

(A_5) For each $r > 0$ there exists a $M(r) > 0$ such that $D(F(t, x(t), \phi), \hat{0}) \le M(r)$ for $D(x, (t), \hat{0}) \le r, D(\phi, \hat{0}) \le r$ for $t \in R, \hat{0} \in E^n$ and for any $\phi \in \Gamma$.

(A_6) In the condition (A_4), and for any fuzzy mappings $x(t)$, $y(t)$ continuous on R there is $p > M/r$, (3.1) is also valid.

Remark 3.1 : The condition (A_6) contains the condition (A_4).

We consider the fuzzy functional differential equations with infinite time delays

$$x'(t) = F(t, x(t), x_t). \qquad \text{... (3.2)}$$

Definition 3.1 — A solution $x(t, t_0, \phi_1)$, which satisfies $x_{t_0} = \phi_1$ with $\phi_1 \in \Gamma$ and $D(x(t, t_0, \phi_1), \hat{0}) \le M$ for $t \ge t_0, \hat{0} \in E^n$ is uniformly stable if, for each $\varepsilon > 0$ and each $t_0 \ge 0$, there exists a positive number $\delta = \delta(\varepsilon)$ (independent of t_0) such that $D(x(t, t_0, \phi_1), y(t, t_0, \phi_2)) < \varepsilon$, whenever $D(\phi_1, \phi_2) < \delta$ and $t \ge t_0$, where $y(t, t_0, \phi_2)$, which satisfies $y_{t_0} = \phi_2$ with $\phi_2 \in \Gamma$, is any solution of (3.2) for $t \ge t_0$.

Theorem 3.1 — Suppose that the fuzzy mapping $F: R \times E^n \times \Gamma \to E^n$ satisfies $(A_2) - (A_5)$. Then (3.2) has only one solution $\bar{x}(t)$ with $D(\bar{x}(t), \hat{0}) \le r$ for $t \in R$ and for any $\hat{0} \in E^n$.

PROOF : We write (3.2) in the following form :

$$x'(t) = F(t, x(t), x_t) = -\frac{1}{h} x(t) + g(t, x(t), x_t) + F(t, 0, 0), \quad \text{... (3.3)}$$

where $g(t, x(t), x_t) = \frac{1}{h} x(t) + F(t, x(t), x_t) - F(t, 0, 0)$. Then there is $g(t, 0, 0) = 0$ and it follows from (A_3) and (A_4) that $g(t, x(t), x_t)$ satisfies following condition :

(A_7) $g(t, x(t), x_t)$ is uniformly continuous on R and satisfies

$$D(g(t, x(t), x_t),\ g(t, y(t), y_t) \le \frac{1-ph}{h} D(x_t, y_t) \quad \text{... (3.4)}$$

for $t \in R$ and fuzzy mappings $x(t)$, $y(t)$ uniformly continuous on R such that

$$D(x(t), \hat{0}) \le r, D(y(t), \hat{0}) \le r \text{ for } t \in R, \hat{0} \in E^n.$$

(I) We prove that (3.2) has a solution $\bar{x}(t)$ with $D(\bar{x}(t), \hat{0}) \le M/p$ for $t \in R, \hat{0} \in E^n$.

We first construct the mapping sequences as follows :

$$x^0(t) = \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) F(s, 0, 0) ds, \text{ for } t \in R, \quad \text{... (3.5)}$$

$$x^{m+1}(t) = x^0(t) + \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) g(s, x^m(s), x_s^m)\, ds, \text{ for } t \in R \quad \text{... (3.6)}$$

when $m = 0, 1, 2, 3, \ldots$.

(i) We prove that $x^m(t)$, $(m = 0, 1, 2, \ldots)$ are uniformly continuous and $D(x^m(t), \hat{0}) < M/p$ for $t \in R, \hat{0} \in E^n$. From (A_2) and (A_3), there is

$$\left.\begin{aligned} D(x^0(t), \hat{0}) &\le \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) D(F(s, 0, 0), \hat{0})\, ds \\ &\le M \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) ds \\ &= Mh < \frac{M}{p} \qquad \text{for } t \in R, \hat{0} \in E^n. \end{aligned}\right] \quad \text{.... (3.7)}$$

And it follows from (A_3) and (3.5) that $x^0(t)$ is uniformly continuous on R. Using (A_7), (3.6) and (3.7), we have

$$\begin{aligned} D(x^1(t),\hat{0}) &\le D(x^0(t),\hat{0}) + \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) D(g(s,x^0(s),x_s^0)\,\hat{0})\,ds \\ &\le Mh + \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right)\frac{(1-ph)}{h} D(x_s^0,\hat{0})\,ds \\ &\le Mh + (1-ph)\,M \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) ds \\ &= Mh(1+(1-ph)) \quad \text{for} \quad t\in R, \hat{0}\in E^n \end{aligned} \qquad \text{... (3.8)}$$

Since $0 < 1 - ph < 1$, there is

$$\frac{1}{ph} = \frac{1}{1-(1-ph)} = \sum_{j=0}^{\infty} (1-ph)^j. \qquad \text{... (3.9)}$$

It follows from (3.8) and (3.9), we see that

$$D(x^1(t),\hat{0}) \le Mh[1+(1-ph)] < M/p \quad \text{for} \quad r\in R, \hat{0}\in E^n. \qquad \text{... (3.10)}$$

From (A_7), (3.6)-(3.8) and the uniform continuity of $x^0(t)$ on R, we see easily that $x^1(t)$ is uniformly continuous on R.

In general, for any natural number k, we can assume inductively that

$$D(x^k(t),\hat{0}) \le Mh \sum_{j=0}^{k} (1-ph)^j < \frac{M}{p} \quad \text{for } t\in R, \hat{0}\in E^n, \qquad \text{... (3.11)}$$

and $x^k(t)$ is unifomly continuous on R. Then from (3.4), (3.6), (3.7) and (3.11), there is

$$\begin{aligned} D(x^{k+1}(t),\hat{0}) &\le D(x^0(t),\hat{0}) + \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) D(g(s,x^k(s),x_s^k),\hat{0})\,ds \\ &\le Mh + \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right)\frac{(1-ph)}{h} D(x_s^k,\hat{0})\,ds \\ &\le Mh + (1-ph)\,M \sum_{j=0}^{k} (1-ph)^j \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) ds \\ &= Mh \sum_{j=0}^{k+1} (1-ph)^j < \frac{M}{p} \quad \text{for } t\in R, \hat{0}\in E^n \end{aligned} \qquad \text{... (3.12)}$$

Because of (A_7), (3.6) and the uniform continuity of $x^0(t)$ and $x^k(t)$ on R, $x^{k+1}(t)$ is uniformly continuous on R. Therefore by induction, for any natural number m, there is

$$D(x^m(t), \hat{0}) \leq Mh \sum_{j=0}^{m} (1 - ph)^j < \frac{M}{p} \text{ for } t \in R, \hat{0} \in E^n. \qquad \text{... (3.13)}$$

and $x^m(t)$ is uniformly continuous on R.

(*ii*) We prove that $\{x^m(t)\}$ is uniformly convergent on R and its limit mapping $\bar{x}(t)$ is uniformly continuous and satisfies $D(\bar{x}(t), \hat{0}) \leq M/p$ for $t \in R, \hat{0} \in E^n$. Let

$$L_{m+1} = \sup \{D(x^{m+1}(t), x^m(t)) : t \in R\}, \ m = 0, 1, 2, \dots . \qquad \text{... (3.14)}$$

Then from (3.4), (3.6), (3.14), and (A_7) and (i) we have

$$\left.\begin{aligned} D(x^{m+1}(t), x^m(t)) &\leq \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) D(g(s, x^m(s), x_s^m), g(s, x^{m-1}(s), x_s^{m-1}))\, ds \\ &\leq \frac{1-ph}{h} \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) D(x_s^m, x_s^{m-1})\, ds \\ &\leq \frac{(1-ph) L_m}{h} \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) ds \\ &= (1-ph) L_m \quad \text{for } t \in R. \end{aligned}\right] \qquad \text{... (3.15)}$$

Thus, we get

$$L_{m+1} \leq (1 - ph) L_m, \ m = 0, 1, 2, \dots .$$

Since $0 < 1 - ph < 1$ and (E^n, D) is complete metric space, the sequence $\{x^m(t)\}$ of fuzzy mappings is uniformly convergent on R. Let $D(x^m(t), \bar{x}(t)) \to 0$ uniformly in $t \in R$ as $m \to \infty$. Since $x^m(t), m = 0, 1, 2, \dots,$ are uniformly continuous on R and $D(x^m(t), \hat{0}) \leq M/p$ for $t \in R, \hat{0} \in E^n$, hence $\bar{x}(t)$ is uniformly continuous on R and $D(\bar{x}(t), \hat{0}) < M/p$ for $t \in R, \hat{0} \in E^n$.

(*iii*) We prove that

$$D\left(\int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) g(s, x^m(s), x_s^m)\, ds, \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) g(s, \bar{x}(s), \bar{x}_s)\, ds\right) \to 0$$

uniformly in $t \in R$ when $m \to \infty$. Since $D(x^m(t), \bar{x}(t)) \to 0$ uniformly on R when $m \to \infty$, for each $\varepsilon > 0$, there is a natural number $N = N(\varepsilon)$ sufficiently large such that

$$D(x^m(t), \bar{x}(t)) < \varepsilon \text{ for } t \in R,$$

when $m \geq N$. Thus, if $m \geq N$, using (A_7), we have

$$D\left(\int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) g(s, x^m(s), x_s^m)\, ds, \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) g(s, \bar{x}(s), \bar{x}_s)\, ds\right)$$

$$\leq \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) D(g(s, x^m(s), x_s^m), g(s, \bar{x}(s), \bar{x}_s)\, ds$$

$$\leq \frac{(1-ph)}{h} \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) D(x_s^m, \bar{x}_s)\, ds$$

$$\leq \frac{(1-ph)\varepsilon}{h} \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) ds$$

$$= (1-ph)\,\varepsilon < \varepsilon$$

for all $t \in R$, i.e.,

$$D\left(\int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) g(s, x^m(s), x_s^m)\, ds, \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) g(s, \bar{x}(s), \bar{x}_s)\, ds\right) \to 0$$

uniformly for $t \in R$, as $m \to \infty$.
Now, taking its limit from the both sides of (3.6), we obtain

$$\bar{x}(t) = x^0(t) + \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) g(s, \bar{x}(s), \bar{x}_s)\, ds \text{ for } t \in R. \qquad \text{... (3.16)}$$

Then, from the right side of (3.16), it is easy to see that $\bar{x}(t)$ is continuously differentiable on R. Immediately, differentiating the both side of (3.16), we have $\bar{x}'(t) = F(t, \bar{x}(t), \bar{x}_t)$ for $t \in R$. Therefore, $\bar{x}(t)$ is a bounded solution of (3.2) such that $D(\bar{x}(t), \hat{0}) \leq M/p$ for $t \in R, \hat{0} \in E^n$.

(II) We prove that $\bar{x}(t)$ is a unique bounded solution of (3.2) such that $D(\bar{x}(t), \hat{0}) \leq r$ for $t \in R, \hat{0} \in E^n$.

In fact, if this conclusion is not valid, then there is another bounded solution $y(t)$ of (3.2) such that $D(y(t), \hat{0}) \leq r$ for $t \in R, \hat{0} \in E^n$ and $y(t) \neq \bar{x}(t)$. It follows from (A_5) that $y(t)$ is uniformly continuous on R. Since

$$\bar{x}'(t) - y'(t) = -\frac{1}{h}[\bar{x}(t) - y(t)] + [g(t, \bar{x}(t), \bar{x}_t) - g(t, y(t), y_t)],$$

we get

$$\bar{x}(t) - y(t) = \exp\left(-\frac{1}{h}(t-t_0)\right)[\bar{x}(t_0) - y(t_0)]$$

$$+ \int_{t_0}^{t} \exp\left(-\frac{1}{h}(t-s)\right)[g(t, \bar{x}(s), \bar{x}_s) - g(s, y(s), y_s)]\, ds, \text{ for } t \geq t_0. \qquad \ldots (3.17)$$

Using (A_7), we have

$$D(\bar{x}(t), y(t)) \leq \exp\left(-\frac{1}{h}(t-t_0)\right) D(\bar{x}(t_0), y(t_0))$$

$$+ \frac{(1-ph)}{h} \int_{t_0}^{t} \exp\left(-\frac{1}{h}(t-s)\right) D(\bar{x}_s, y_s)\, ds \text{ for } t \geq t_0. \qquad \ldots (3.18)$$

Let $c_1 = \sup\{D(\bar{x}(t), y(t)) : t \in R\}$. There are $c_1 > 0$ and $t_1 \in R$ such that

$$D(\bar{x}(t_1), y(t_1)) \geq c_1\left(1 - \frac{ph}{4}\right) \qquad \ldots (3.19)$$

Taking $t_0 = t_1 - T$, where $T > 0$ is sufficiently large with

$$\exp\left(-\frac{T}{h}\right) \leq \frac{ph}{2}. \qquad \ldots (3.20)$$

From (3.18)-(3.20), we obtain

$$c_1\left(1 - \frac{ph}{4}\right) \leq D(\bar{x}(t_1), y(t_1))$$

$$\leq \exp\left(-\frac{T}{h}\right) c_1 + (1-ph)\, c_1$$

$$\leq \frac{phc_1}{2} + (1-ph)c_1$$

$$= \left(1 - \frac{ph}{2}\right) c_1,$$

which is a contradiction because of $0 < 1 - ph < 1$ and $c_1 > 0$. Therefore $\bar{x}(t)$ is a unique bounded solution of (3.2) such that $D(\bar{x}(t), \hat{0}) \leq r$ for $t \in R, \hat{0} \in E^n$. It completes the proof.

Theorem 3.2 — *Suppose that the fuzzy mapping $F : R \times E^n \times \Gamma \to E^n$ has (A_2), (A_3), (A_5) and (A_6). Then (3.2) has only one bounded solution $\bar{x}(t)$ which is uniformly stable and satisfies $D(\bar{x}(t), \hat{0}) \leq r$ for $t \in R, \hat{0} \in E^n$.*

PROOF : From Remark 3.1 and Theorem 3.1, we can see that it suffices to prove that $\bar{x}(t)$ is uniformly stable. In fact, for each $\varepsilon > 0$ (especially $\varepsilon < r - M/p$) and each $t_0 \geq 0$, taking a positive number $\delta = \delta(\varepsilon) = \min\{(r - M/p)/2, ph\,\varepsilon/2\}$ (obviously, $\delta(\varepsilon)$ is independent of t_0), we see that $D(\bar{x}_{t_0}, \phi) < \delta$, then we have

$$D(\bar{x}(t), y(t)) < \varepsilon, \text{ for } t \geq t_0 \qquad \ldots (3.21)$$

where $y(t) = y(t, t_0, \phi)$, which satisfies $y_{t_0} = \phi$ with $\phi \in \Gamma$, is any solution of (3.2) for $t \geq t_0$. Otherwise, if (3.21) is not valid, there is $t_1 > t_0$ such that

$$D(\bar{x}(t_1), y(t_1)) = \varepsilon \qquad \ldots (3.22)$$

and

$$D(\bar{x}(t), y(t)) < \varepsilon \text{ for } t_0 < t < t_1. \qquad \ldots (3.23)$$

Thus we get

$$D(\bar{x}_t, y_t) \leq \varepsilon \text{ for } t \leq t_1. \qquad \ldots (3.24)$$

Since

$$\begin{aligned} \bar{x}'(t) - y'(t) &= F(t, \bar{x}(t), \bar{x}_t) - F(t, y(t), y_t) \\ &= -\frac{1}{h}[\bar{x}(t) - y(t)] + [g_1(t, \bar{x}(t), \bar{x}_t) - g_1(t, y(t), y_t)] \text{ for } t \geq t_0, \end{aligned} \qquad \ldots (3.25)$$

where $g_1(t, x(t), x_t) = \frac{1}{h}x(t) + F(t, x(t), x_t)$ and from (A_6), we have

$$D(g_1(t, x(t), x_t), g_1(t, y(t), y_t)) \leq \frac{1 - ph}{h} D(x_t, y_t) \qquad \ldots (3.26)$$

for any continuous fuzzy mappings $x(t)$, $y(t)$ with $D(x(t), \hat{0}), \leq r$ and $D(y(t), \hat{0}) \leq r$ for $t \in R, \hat{0} \in E^n$. It follows from (3.25) that

$$\begin{aligned} \bar{x}(t_1) - y(t_1) &= \exp\left(-\frac{1}{h}(t_1 - t_0)\right)[\bar{x}(t_0) - y(t_0)] \\ &+ \int_{t_0}^{t_1} \exp\left(-\frac{1}{h}(t_1 - s)\right)[g_1(s, \bar{x}(s), \bar{x}_s) - g_1(s, y(s), y_s)]\, ds. \end{aligned} \qquad \ldots (3.27)$$

Using (3.22)-(3.24), (3.26) and (3.27), we can obtain

$$\varepsilon = D(\bar{x}(t_1), y(t_1)) \leq \exp\left(-\frac{1}{h}(t_1 - t_0)\right) D(\bar{x}(t_0), y(t_0))$$

$$+ \int_{t_0}^{t_1} \exp\left(-\frac{1}{h}(t_1 - s) \right) D(g_1(s, \bar{x}(s), \bar{x}_s), g_1(s, y(s), y_s))\, ds$$

$$\leq \exp\left(-\frac{1}{h}(t_1 - t_0) \right)\delta + \frac{1-ph}{h} \int_{t_0}^{t_1} \exp\left(-\frac{1}{h}(t_1 - s) \right) D(\bar{x}_s, y_s)\, ds$$

$$\leq \delta + \frac{(1-ph)\varepsilon}{h} \int_{t_0}^{t_1} \exp\left(-\frac{1}{h}(t_1 - s) \right) ds$$

$$\leq \delta + (1-ph)\,\varepsilon \leq \frac{ph\varepsilon}{2} + (1-ph)\,\varepsilon$$

$$= \left(1 - \frac{ph}{2} \right)\varepsilon < \varepsilon,$$

which is a contradiction. Therefore, $\bar{x}(t)$ is uniformly stable.

***Theorem** 3.3* — *If the fuzzy mapping $F : R \times E^n \times \Gamma \to E^n$ is periodic in t with period T independent of (t, ϕ) and if the fuzzy mapping $F : R \times E^n \times \Gamma \to E^n$ satisfies conditions $(A_2) - (A_4)$, then (3.2) has only one T-periodic solution $\bar{x}(t)$ with $D(\bar{x}(t), \hat{0}) \leq r$ for $t \in R, \hat{0} \in E^n$.*

PROOF : Because a continuous T-periodic solution of (3.2) is uniformly continuous on R, the condition (A_5) is not need. From Theorem 3.1, obviously, it suffices to prove that $\bar{x}(t)$ is a T-periodic solution of (3.2). Using (3.16) and the conditions of this theorem, we obtain

$$\left.\begin{aligned}
\bar{x}(t+T) &= \int_{-\infty}^{t+T} \exp\left(-\frac{1}{h}(t+T-s) \right)[g(s, \bar{x}(s), \bar{x}_s) + F(s, 0, 0)]\, ds \\
&= \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-u) \right)[g(u+T, \bar{x}(u+T), \bar{x}_{u+T}) + F(u+T, 0, 0)]\, du \\
&= \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-u) \right)[g(u, \bar{x}(u+T), \bar{x}_{u+T}) + F(u, 0, 0)]\, du \\
&= \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s) \right)[g(s, \bar{x}(s+T), \bar{x}_{s+T}) + F(s, 0, 0)]\, ds
\end{aligned}\right] \quad \text{... (3.28)}$$

Let $L = \sup \{D(\bar{x}(t), \bar{x}(t+T)) : t \in R\}$. Then from (3.16), (3.28) and (A_7), we have

$$D(\bar{x}(t), \bar{x}(t+T)) \leq \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s) \right) D(g(s, \bar{x}(s), \bar{x}_s), g(s, \bar{x}(s+T), \bar{x}_{s+T}))\, ds$$

$$\leq \frac{(1-ph)L}{h} \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right) ds$$

$$= (1-ph)\,L,$$

i.e. $L \leq (1-ph)L$. It follows from $0 < 1 - ph < 1$ that $L = 0$. Therefore, $\bar{x}(t)$ is a T-periodic solution of (3.2).

***Theorem** 3.4 — Suppose that the fuzzy mapping $F : R \times E^n \times \Gamma \to E^n$ has $(A_1) - (A_4)$. Then (3.2) has unique almost periodic solution $\bar{x}(t)$ with $D(\bar{x}(t), \hat{0}) \leq r$ for $t \in R, \hat{0} \in E^n$.*

PROOF : Because an almost periodic solution of (3.2) is uniformly continuous on R, the condition (A_5) is not needed. From Theorem 3.1, obviously, it suffices to prove that $\bar{x}(t)$ is an almost periodic solution of (3.2). Using (3.16), we have

$$\bar{x}(t+\tau) = \int_{-\infty}^{t+\tau} \exp\left(-\frac{1}{h}(t+\tau-s)\right)[g(s, \bar{x}(s), \bar{x}_s) + F(s, 0, 0)]\, ds$$

$$= \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-u)\right)[g(u+\tau, \bar{x}(u+\tau), \bar{x}_{u+\tau}) + F(u+\tau, 0, 0)]\, du$$

$$= \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right)[g(s+\tau, \bar{x}(s+\tau), \bar{x}_{s+\tau}) + F(u+\tau, 0, 0)]\, ds$$

for all $t, \tau \in R$. It follows from (3.16) and (3.29) that

$$\bar{x}(t) - \bar{x}(t+\tau) = \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right)\frac{1}{h}[\bar{x}(s) - \bar{x}(s+\tau) + h(F(s, \bar{x}(s), \bar{x}_s)$$

$$- F(s+\tau, \bar{x}(s+\tau), \bar{x}_{s+\tau}))]\, ds \qquad \text{... (3.30)}$$

$$= \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right)\left[\frac{1}{h}\left[\bar{x}(s) - \bar{x}(s+\tau) + h(F(s, \bar{x}(s), \bar{x}_s) - F(s, \bar{x}(s+\tau), \bar{x}_{s+\tau}))\right]\right.$$

$$\left. + [F(s, \bar{x}(s+\tau), \bar{x}_{s+\tau}) - F(s+\tau, \bar{x}(s+\tau), \bar{x}_{s+\tau})]\right] ds$$

for all $t \in R$. By (A_1), for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$ such that every interval of R of length $l(\varepsilon)$ contains a $\tau = \tau(\varepsilon)$ with

$$D(F(t, \bar{x}(t+\tau), \bar{x}_{t+\tau}) - F(t+\tau, \bar{x}(t+\tau), \bar{x}_{t+\tau})) < \varepsilon \qquad \text{... (3.31)}$$

for all $t \in R$. Let $L_0 = \sup\{D(\bar{x}(t), \bar{x}_{t+\tau}) : t \in R, \tau = \tau(\varepsilon)\}$. Thus for such a τ, it follows from (A_7), (2.30) and (2.31) that

$$D(\overline{x}(t), \overline{x}(t+\tau)) \leq \int_{-\infty}^{t} \exp\left(-\frac{1}{h}(t-s)\right)\left[\frac{(1-ph)}{h} D(\overline{x}_s, \overline{x}_{s+\tau}) + \varepsilon\right] ds$$

$$\leq (1-ph)\, L_0 + h\varepsilon,$$

i.e., $L_0 \leq (1-ph)\, L_0 + h\varepsilon$. Therefore, there is $L_0 \leq \varepsilon/p$. Thus τ is an ε/p-translation number for $\overline{x}(t)$, and since $\varepsilon > 0$ is arbitrary, $\overline{x}(t)$ is almost periodic fuzzy mapping F. The proof is complete.

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