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ON A VARIATIONAL-TYPE INEQUALITY AND ITS COMPLEMENTARITY PROBLEM IN HAUSDORFF TOPOLOGICAL VECTOR SPACES

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In this paper we discuss the existence of the solution of the variational-type inequality problem in Hausdorff topological vector spaces. This problem was originally introduced by Behera and Panda in the setting of reflexive real Banach spaces.

Key Words : Topological Vector Spaces; Locally Convex Spaces; Variational Inequalities; KKM-maps

1. INTRODUCTION

Let (X, X^*) be a dual system of locally convex spaces or Banach spaces over the field of real numbers and K a nonempty subset of X. Let the value of $f \in X^*$ at $x \in X$ be denoted by (f, x). Let $T : K \to X^*$ be an operator, possibly nonlinear.

The variational inequality problem associated to T and K is to find $x_0 \in K$ such that

$$(Tx_0, y - x_0) \geq 0 \qquad \dots (1)$$

for all $y \in K$.

If k is a cone, the complementarity problem associated to T and K is to find $x_0 \in K$ such that $Tx_0 \in K^*$ and

$$(Tx_0, x_0) = 0$$

where K^* is the polar of K defined by

$$K^* = \{f \in X^* : (f, x) \geq 0 \text{ for all } x \in K\}.$$

The variational inequality and complementarity problems are intensely studied since they have interesting applications in areas such as optimization theory, game theory, engineering, structural mechanics, elasticity theory, lubrication theory, equilibrium theory on networks, flow in porous media and so on.

The existence of solution of these problem under different assumptions are very widely found in literature. The interested readers are referred to Giannessi[5], Isac[7] and Kinderlehrer and Stampacchia[8].

In recent years several extensions and variations of these problems have been proposed and studied by many authors, some of which are found in the works of Behera and Panda[1,2,3], Ding[4], Siddiqi, Ansari and Ahmad[9] and Siddiqi, Khaliq and Ansari[10].

Behera and Panda[1] introduced the following problems :

Problem 1 — Given $z \in K$, find $x_0 \in K$ such that

$$\left(T\left(\frac{z+x_0}{2}\right), y - x_0\right) \geq 0 \qquad \ldots (2)$$

for all $y \in K$. Inequality (2) is called the variational type inequality.

Problem 2 — Let K be a cone. Given $z \in K$, find $x_0 \in K$ such that

$$T\left(\frac{z+x_0}{2}\right) \in K^*$$

and

$$\left(T\left(\frac{z+x_0}{2}\right), x_0\right) = 0,$$

which is the complementarity problem corresponding to the variational type inequality (2).

It is clear that inequality (2) is not a generalization of inequality (1); but one can easily see that any fixed point of the correspondence $z \mapsto x_0$ in (2) is a solution of inequality (1). This shows that the study of inequality (2) is also equally important as that of (1).

Behera and Panda[1] proved the following theorems on the existence solutions of Problems 1 and 2 in reflexive real Banach spaces.

***Theorem** 1.1 — Let K be a closed and convex subset in a reflexive real Banach space X, with $0 \in K$ and let X^* be the dual of X. Let $T : K \to X^*$ be a monotone and hemicontinuous map. Then for each $z \in K$, there exists $x_0 \in K$ such that*

$$\left(T\left(\frac{z+x_0}{2}\right), y - x_0\right) \geq 0$$

for all $y \in K$, under each of the following conditions :

(a) For each $y \in K$, there exists a real number $L > 0$ such that

$$(Tx,\ y - x) < 0$$

whenever

$$\| x \| \geq L.$$

(b) K is bounded.

***Theorem** 1.2 — Let X be a reflexive real Banach space with dual X^* and K a closed and convex cone in X, with vertex at the origin and polar K^*. Let $T: K \to X^*$ be a monotone and hemicontinuous map. Furthermore assume that for each $y \in K$, there exists a real number $L > 0$ such that*

$$(Tx,\ y - x) < 0$$

whenever $\| x \| \geq L$. Then for each $z \in K$, there exists $x_0 \in K$ such that

$$T\left(\frac{z + x_0}{2}\right) \in K^*$$

and

$$\left(T\left(\frac{z + x_0}{2}\right), x_0\right) = 0.$$

In this paper we study the existence of solutions of Problems 1 and 2 in the setting of Hausdorff topological vector spaces.

The following results[6,7] will play a crucial role in the proof of the main results of this paper.

Definition 1.3 — A point-to-set map $F: K \to 2^X$ is called a *KKM*-map if for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of K,

$$\text{Conv}\ (\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} F(x_i),$$

where Conv(A) denotes the convex hull of any subset A of X.

***Theorem** 1.4 — Let K be an arbitrary nonempty subset of a Hausdorff topological vector space and $F: K \to 2^X$ a KKM-map. If F(x) is closed for all $x \in K$ and is compact for at least one $x \in K$ then*

$$\bigcap_{x \in K} F(x) \neq \phi.$$

2. Existence of Solution

The following theorems on the existence of solution of the Problems 1 and 2 are the main results of this paper.

***Theorem** 2.1 — Let K be a nonempty convex set in a Hausdorff topological vector space X and let X^* be the dual of X. Let $T: K \to X^*$ be an operator such that*

(i) the map $x \mapsto (Tx, y-x)$ is upper semicontinuous for each $y \in K$ and

(ii) there exists a nonempty compact and convex set $L \subset K$ such that for every $x \in K-L$ and $z \in K$, there exists $u \in L$ such that

$$\left(T\left(\frac{z+x}{2}\right), u-x\right) < 0.$$

Then Problem 1 has a solution.

PROOF : Let $z \in K$ be given. Define a set valued map $F: K \to 2^X$ as

$$F(y) = \left\{x \in L : \left(T\left(\frac{z+x}{2}\right), y-x\right) \geq 0\right\}.$$

It is clear that (2) has a solution if

$$\bigcap_{y \in K} F(y) \neq \phi. \quad \ldots (3)$$

Since by hypothesis (i), the map $x \mapsto (Tx, y-x)$ is upper semicontinuous,

$$\left(T\left(\frac{z+x}{2}\right), y-x\right) = \frac{1}{2}\left(T\left(\frac{z+x}{2}\right), \left(\frac{z+y}{2}\right) - \left(\frac{z+x}{2}\right)\right)$$

for each $y \in K$ and the map $x \mapsto \frac{z+x}{2}$ is continuous, it follows that the map

$$x \mapsto \left(T\left(\frac{z+x}{2}\right), y-x\right)$$

is also upper semicontinuous for each $y \in K$. Thus the set

$$G(y) = \left\{x \in K : \left(T\left(\frac{z+x}{2}\right), y-x\right) \geq 0\right\}$$

is closed for each $y \in K$, and since $F(y) = G(y) \cap L$, and L is compact, it follows that $F(y)$ is also compact for each $y \in K$. Thus for the fulfillment of (3), it is sufficient to prove that the family $\{F(y) : y \in K\}$ has the finite intersection property.

Let $y_1, y_2, \ldots, y_n$ be arbitrary elements of K and let

$$H = \text{Conv}\ (L \cup \{y_1, y_2, \ldots, y_n\}).$$

It is clear that H is a compact convex subset of K. Let us define another point-to-set map $E: H \to 2^X$ as

$$E(y) = \left\{ x \in H : \left(T\left(\frac{z+x}{2} \right), y - x \right) \geq 0 \right\}.$$

It is clear that $E(y)$ is nonempty for each $y \in H$, (since $y \in E(y)$) and $E(y)$ is a compact subset of K for each $y \in H$.

We next prove that E is a KKM-map. If we assume to the contrary that E is not a KKM-map, then there exist

$$\{u_1, u_2, \ldots, u_n\} \subset H,$$

$$a_i \geq 0, \text{ with } \sum_{i=1}^{n} a_i = 1$$

such that

$$w = \sum_{i=1}^{n} a_i u_i \notin \bigcup_{j=1}^{n} E(u_j).$$

Thus $w \notin E(u_j)$ for every $j \in \{1,2, \ldots, n\}$, which is equivalent to the fact that

$$\left(T\left(\frac{z+w}{2} \right), u_j - w \right) < 0.$$

Hence,

$$0 = \left(T\left(\frac{z+w}{2} \right), w - w \right)$$

$$= \sum_{i=1}^{n} a_i \left(T\left(\frac{z+w}{2} \right), u_j - w \right) < 0,$$

which is a contradiction.

Now by Theorem 1.4, we have

$$\bigcap_{y \in H} E(y) \neq \phi.$$

Thus there exists $u_0 \in H$ such that

$$\left(T\left(\frac{z+u_0}{2} \right), y - u_0 \right) \geq 0 \qquad \ldots (4)$$

for all $y \in H$. In fact $u_0 \in L$. If we assume that $u_0 \notin L$ (i.e., $u_0 \in H - L \subset K - L$), then by hypothesis (ii) there exists $u \in L$ such that

$$\left(T\left(\frac{z+u_0}{2}\right),\ u-u_0\right)<0$$

which contradicts (4) when $y = u$.

Thus, $u_0 \in L$ and in particular $u_0 \in F(y_i)$ for each i; i.e.,

$$u_0 \in \bigcap_{i=1}^{n} F(y_i),$$

which proves that the family $\{F(y) : y \in K\}$ has finite intersection property.

Thus there exists $x_0 \in K$ such that

$$\left(T\left(\frac{z+x_0}{2}\right), y-x_0\right) \geq 0$$

for all $y \in K$. Since $x_0 \in F(y)$ for each $y \in K$ and $F(y) \subset L$, it follows that $x_0 \in L$. This completes the proof of Theorem 2.1.

The following Theorems on the existence of solutions of Problem-2 are direct consequences of Theorem 2.1.

Theorem 2.2 — *Let* (X, X^*), *be a dual system of locally convex spaces,* $K \subset X$ *a convex cone with vertex at the origin and polar* K^* *and* $T : K \to X^*$ *be an operator such that*

(i) for each $y \in K$, *the map* $x \mapsto (Tx, y-x)$ *is upper semi-continuous on* K, *and*

(ii) there exists a nonempty compact convex subset L *of* X *such that for every* $x \in K-L$. *there exists,* $u \in L$ *such that*

$$\left(T\left(\frac{z+x}{2}\right), u-x\right) < 0.$$

Then there exists $x_0 \in L$ *such that* $T\left(\frac{z+x_0}{2}\right) \in K^*$ *and*

$$\left(T\left(\frac{z+x_0}{2}\right), x_0\right) = 0.$$

PROOF : By Theorem 2.1, there exists $x_0 \in L$ such that

$$\left(T\left(\frac{z+x_0}{2}\right), y-x_0\right) \geq 0 \qquad \text{... (5)}$$

for all $y \in K$. Since K is a cone, $2x_0 \in K$. Putting $y = 0$ and $y = 2x_0$ successively in (6) we get

$$\left(T\left(\frac{z+x_0}{2}\right), x_0\right) \geq 0, \left(T\left(\frac{z+x_0}{2}\right), x_0\right) \leq 0.$$

Combining these inequalities we get

$$\left(T\left(\frac{z+x_0}{2} \right), x_0 \right) = 0. \qquad \ldots (6)$$

Substituting (6) in (5) we obtain

$$\left(T\left(\frac{z+x_0}{2} \right), y \right) \geq 0$$

for all $y \in K$, which proves that

$$T\left(\frac{z+x_0}{2} \right) \in K^*.$$

This completes the proof.

***Theorem** 2.3 — Let (X, X^*) be a dual system of locally convex spaces, $K \subset X$ a nonempty convex cone and $T : K \to X^*$ an operator such that the map $(x, y) \mapsto (Tx, y)$ is continuous on $K \times K$. Assume that there exists a nonempty compact convex subset L or K such that for every $z \in K$ and $x \in K - L$ there exists $u \in L$ such that*

$$\left(T\left(\frac{z+x}{2} \right), u - x \right) < 0.$$

Then there exists $x_0 \in L$ such that

$$T\left(\frac{z+x_0}{2} \right) \in K^*$$

and

$$\left(T\left(\frac{z+x_0}{2} \right), x_0 \right) = 0.$$

PROOF : Since continuity of $(x, y) \mapsto (Tx, y)$ implies the continuity of $x \mapsto (Tx, y - x)$ for fixed y, and since continuity implies upper semi-continuity, the proof of this theorem follows directly from the proof of Theorem 2.2.

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