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COSMOLOGICAL MODELS WITH VARIABLE COSMOLOGICAL CONSTANT AND GRAVITATIONAL CONSTANT

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Following the gravitational theory proposed by Bermann, homogeneous and isotropic cosmological models with variable cosmological constant and gravitational constant have been constructed which satisfy the present day observational data and the initial condition as proposed by Sivaram *et al.*,[23] with additional requirement that $8\pi G\rho = \alpha\Lambda c^2$.

Key Words : Cosmological Models; Constants - Variable Cosmological and Gravitational

1. INTRODUCTION

The cosmological term Λg_{ij} has been introduced by Einstein into his field equations to construct a static model of the universe. Later on, it has been observed that the universe is expanding and therefore the Cosmological term has not been of much significance, recently it has been shown that Λ is not a constant but it is a variable quantity[1,2,3,4]. The same is true for the gravitational Constant G[4,5]. To incorporate the variable gravitational Constant G and the variable Cosmological term Λ into the Einstein field equations, a method has been proposed by Berman[4] on the basis of the conservation law, where the field equation are expressed as

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi G T_{ij} + \Lambda g_{ij} \quad \text{... (1)}$$

and the conservation law is expressed as : $\quad T^{ij}_{ji} = 0$ .. (2)

which on applying to (1), we have

$$8\pi G_{,i} T^{ij} + \Lambda_{,i} g^{ij} = 0. \quad \text{... (3)}$$

This shows that Λ and G both vary simultaneously. Berman[4] and Rahman[6] have constructed homogenerous and isotropic Cosmological models with the help of the equations (1), (2) and (3) by assuming the variations of $G\rho$ and Λ as :

$$G\rho = At^{-2}, \ \Lambda = Bt^{-2} \qquad \text{... (4)}$$

Here we study the Cosmological models by assuming

$$G\rho = \frac{\alpha\Lambda}{8\pi}, \qquad \text{... (5)}$$

where α is a constant and therefore the variations of $G\rho$ and Λ given in (4) is included in (5), we have obtained the constant α from the present day observational data and the initial conditions which are imposed by the strong gravity theory[7,8] on a Cosmological model.

2. Equations Governing the Cosmological Models

For a homogeneous and isotropic cosmological model the space-time metric is given

$$ds^2 = c^2 dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \qquad \text{... (6)}$$

This describes the space-time geometry of the universe on large scales in comoving coordinates, that is, the content of the universe are on average at rest in the coordinate systems (r, θ, ϕ). The time 't' is the proper time for the particles, which are in this case clusters of galaxies. k is the curvature index, such that $k = 1$ for a closed model, $k = 0$ for a flat model and $k = -1$, for an open model. $R(t)$ is the scale factor.

Let us consider a particle at the origin $r = 0$ and another particle at 'r' them the proper distance 'D' between the two particles at a time 't' is given by

$$D = R(t)\int_0^T \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} R\sinh^{-1} r & (k=-1) \\ Rr & (k=0) \\ R\sin^{-1} r & (k=1) \end{cases}. \qquad \text{... (7)}$$

Therefore, the proper distance 'D' is proportional to the scale '$R(t)$'. The proper velocity 'v' of the particle at 'r' relative to the particle at the origin is obtained by differentiating 'D' w.r. to 't' realining that 'r' remains constant because it is a coming co-ordinate. Thus, we have

$$v = \dot{D} = \dot{R}\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \frac{\dot{R}}{R}D, \qquad \text{... (8)}$$

where a dot on the symbol means differentiation with respect to 't'. This tells us that at any time 't' the speed 'v' is proportional to the proper distance 'D'. Comparing this result with the Hubble's law

$$v = HD,$$

we get

$$H = \frac{\dot{R}}{R} \qquad \text{... (9)}$$

In fact, we cannot stretch the measuring tape between the particles, therefore, the proper distance '*D*' is not a measurable quantity. However, the great advantage of the relativistic formulation is that, it gives relationship between quantities such as red-shifts, apparent magnitudes, number counts etc. which can be measured. The important point is that, they all involve only the scale factor $R(t)$.

Therefore, the scale factor has been treated as a measure of distance. In a similar way $\dot{R}$ and $\ddot{R}$ are treated as the measures of the velocity and the acceleration of the fundamental particles relative to the origin that is, in the space-time metric[6], we consider the comoving coordinate 'r' dimensionless and the scale factor $R(t)$ as having the dimension of length[9].

By taking large scale viewpoint the energy momentum tensor of the content of the universe takes the same form as for a perfect fluid distribution of matter.

$$T^{ij} = \left(\frac{p}{c^2} + \rho \right) u^i u^j - p g^{ij}, \qquad \text{... (10)}$$

where 'p' is the proper pressure, ρ is the proper density and u^i is the four velocity of the fluid particles which are in this case cluster of galaxies. Since the particle is at rest in the coordinate system (r, θ, ϕ), we have

$$u^i = (0, 0, 0, c) \qquad \text{... (11)}$$

following Berman[4], we consider the Einstein field equation as

$$R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = -\frac{8\pi G}{c^4} T_{ij} \qquad \text{... (12)}$$

and the conservation law as

$$T^{ij}_{;j} = 0 \qquad \text{... (13)}$$

which on applying to (12), we get

$$\frac{8\pi}{c^4} G_{,i} T^i_j + \Lambda_{,i} g^i_j = 0 \qquad \text{... (14)}$$

We would like to point out here that instead of the field eqs. (12) Bermann[4] and Rahman[6] consider the field eq. (1). Eq. (1) differs from the equation (12) due to the sign convention of the Curvature tensor, that is, the quantities R_{ij} and R written in (1) are expressed as $-R_{ij}$ and $-R$ in (12). We follow the calculations as given by Lord[10], where the eq. (12) has been used instead of the eq. (1).

Substituting the values from (6), (10) and (11) in (12) we get the following set of equations

$$-2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{kc^2}{R^2} = \frac{8\pi G p}{c^2} - \Lambda c^2 \qquad \text{... (15)}$$

and

$$3\frac{\dot{R}^2}{R^2}+3\frac{kc^2}{R^2}=8\pi G\rho+\Lambda c^2 \qquad \text{... (16)}$$

Differentiating, the equation (16) with respect to 't' we get

$$8\pi(\dot{G}\rho+G\,\dot{\rho})=-\frac{3\dot{R}}{R}\left(\frac{-2\ddot{R}}{R}+\frac{-2\dot{R}^2}{R^2}+\frac{2kc^2}{R^2}\right)-\dot{\Lambda}c^2. \qquad \text{... (17)}$$

Adding (15) and (16) we get

$$-2\frac{\ddot{R}}{R}+2\frac{\dot{R}^2}{R^2}+\frac{2kc^2}{R^2}=8pG\left(\frac{p}{c^2}+\rho\right) \qquad \text{... (18)}$$

and then (17) is simplified as

$$\dot{G}\rho+G\,\dot{\rho}+\frac{3\dot{R}}{R}G\rho=-\frac{3\dot{R}}{R}G\frac{o}{c^2}-\frac{\dot{\Lambda}c^2}{8\pi} \qquad \text{... (19)}$$

Multiplying (19) by R^3, we get

$$c^2R^3\left(\dot{G}\,\rho+\frac{\dot{\Lambda}c^2}{8\pi}\right)+g\left[c^2\frac{d}{dt}(\rho R^3)+3pR^2\dot{R}\right]=0 \qquad \text{... (20)}$$

Eq (13) when simplified with the help of the eqs. (6), (10) and (11) we get the conservation law as

$$c^2\frac{d}{dt}(\rho R^3)+3pR^2\,\dot{R}=0. \qquad \text{... (21)}$$

Equation (21), when substituted in (20), gives

$$\dot{G}\rho+\frac{\dot{\Lambda}c^2}{8\pi}=0 \qquad \text{... (22)}$$

further adding (15) and (16) we get

$$3\ddot{R}=-4\pi GR\left[\frac{3p}{c^2}+\rho-\frac{\Lambda c^2}{4\pi G}\right] \qquad \text{... (23)}$$

Eqs. (16), (21), (22) and (23) are the fundamental equations governing a homogeneous and isotropic cosmological model of the universe. Eqs. (16), (21) and (23) are same as the corresponding equation in general relativity, but, here we have additional eq. (22) due to the variation of G and Λ. We consider the dependence of the pressure in a homogeneous and isotropic model of the universe as

$$p=\beta\rho c^2 \qquad \text{... (24)}$$

For $\beta = 0$, $p = 0$ and then the model is pressureless, also for $\beta = \frac{1}{3}, p = \frac{\rho}{3}$ and then the model is filled with radiation. In usual practice, a model is constructed in two ways either considering $\beta = 0$ or considering $\beta = \frac{1}{3}$. The models constructed with $\beta = 0$ are used to study the universe at present and the models constructed with $\beta = \frac{1}{3}$ are used to study the universe in the past when it was radiation dominated. In this way we do not have a theory how the universe reaches to matter dominated era from the radiation dominated era. To avoid this difficulty, we propose to determine the constant β from the present day observational data and initial conditions discussed in the next section. However, for a positive density and positive pressure of the universes, we have

$$\beta \geq 0. \qquad \text{... (25)}$$

The equality sign in (25) implies that the model is pressureless.

Eqs.(21) and (24) together give the dependence of ρ on R, but this is not sufficient to get the dependence of R on t from eq. (16). For this we need the dependence of G and Λ on R. Also knowing the dependence of ρ on R. Eq. (22) is integrable, when we have an additional equation relating G and Λ. Following Berman[4] and Rahman[6], we consider the dependence of G and Λ as

$$G\rho = \frac{\alpha \Lambda c^2}{8\pi} \qquad \text{... (26)}$$

and thus, the fundamental equation governing the model of the universe are (16), (21), (22), (23), (24) and (26).

3. Present Day Observational Data

The numerical value of the Newtonian Gravitational Constant G is well known and it is of the order of 6.6×10^{-8} c.g.s. units therefore for the construction of the model we propose that

$$t = t_{0,} \; G = G_0 = 6.6 \times 10^{-8} \text{ c.g.s. units} \qquad \text{... (27)}$$

where 't_0' is the time at present epoch to be discussed later on.

On the scale of 10^8 light years and beyond the universe consists of clusters of galaxies[11]. There may be other objects in the universe besides the clusters of galaxies e.g., galaxies that have ceased to radiate, balckholes of all sizes and intergalactic dust etc. But firm experimental evidence for this is lacking. However, a whole range of exotic astronomical objects has been discovered. The most puzzling being the quasi-stellar objects. These appear to be as distant and as bright as galaxies, but they are very compact. Their precise nature is not yet known. Ignoring these complications, the mass density 'ρ_{gal}' due to galaxies has been estimated to be of the order of 3×10^{-31} g/cm^3 [12]. The universe also contains isotropic background radiation. Present day observations[13] shows that the black body temp. is 3° K. The mass equivalent of this radiation[14] is negligible in comparison with the galaxies, because the equivalent mass density of the radiation 'ρ_{rad}' is of the order of 4.5×10^{-34} g/cm^3. Therefore, for the construction of the model we assume the density of the universe of the order of 3×10^{-31} g/cm^3 at present, i.e.,

$$t = t_0,\ \rho = \rho = 3 \times 10^{-31}\ \text{g/cm}^3. \quad \dots (28)$$

An important inference from the Hubble's law (eq. 8) is that, in the past, the galaxies were much closer together than they are today. Therefore, the density of the universe must have been very large in the past. The microwave background radiation[13] suggest a dense radiation which dominated early phase of the universe. As a consequence of this result, it is natural to determine the age of the universe. The microwave background radiation suggests that the age of the universe should be of the order of 6.3×10^{17} sec[12]. Thus for the construction of the cosmological model we consider

$$t = t_0 = 6.3 \times 10^{17}\ \text{sec}. \quad \dots (29)$$

The present value of the scale factor $R(t)$ has been estimated in Fridmann Cosmological models and it is found to be of the order of 10^{28} cm.[15,16]. Therefore, for the construction of the cosmological model, we assume that

$$t = t_0, R = R_0 = 10^{28}\ \text{cm}. \quad \dots (30)$$

A number of Fridmann Cosmological models are constructed by considering non-zero values of the cosmological constant Λ. The cosmological constant Λ has been considered positive as well as negative[17, 18, 19]. At the time of Einstein, the cosmological constant has no direct physical meaning, but today the cosmological constant is assigned a physical meaning. Zel'dovich interprets the cosmological constant as the 'ground-state energy' of the vaccum[1] and hence the numerical values of Λ has been computed to be the order of 10^{-57} cm^{-2}. Therefore, for the construction of the comsological model we assume that

$$t = t_0,\ |\Lambda| = |\Lambda_0| = 10^{-57}\,\text{cm}^{-2} \quad \dots (31)$$

The sign of Λ will be decided on the ground of the physical validity of the model to be discussed in the section 5.

4. Non-Singular Cosmology and the Initial Data

According to the Howking and Penrose[20], singularities are an inherent feature of general relativity and hence, the usual Friedmann cosmological models are singular. They pass through a singular state, when matter collapses to a physically meaningless density. Bekenstein[21] has shown that the Hawking-Penrose energy condition may be avoided if one considers the energy-momentm tensors of a massive scalar field. The work done by Trautmann[22]. Isham Salam and Strathdee[23], Sinha Sivram, Lord and Sudarshan[8, 16] shows that the universe cannot squeeze to a radius smaller than 10^{13} cm with the cirresoinding density of the order of 10^{17} g/cm^3. They argue that the matter in a superdense collapsed state is predominantly composed of Hadrons. Thus, we may take the initial conditions for the construction of a cosmological model as follws

$$t = 0,\ R = R_i = 10^{13}\ \text{cm},\ \rho = \rho_i = 10^{17}\ \text{g/cm}^3,$$

$$\dot{R} = 0,\ \ddot{R} > 0,\ G = G_f = 6.6 \times 10^{30}\ \text{c.g.s. unit} \quad \dots (32)$$

and $\Lambda = \Lambda_f = 10^{28}\ \text{cm}^{-2}$

It would be worthwhile to point out here that Sivaram, Sinha and Lord[8] proposed $-G_f$ instead of G_f at $t = 0$ for the construction a non-singular cosmological model. That is, G is increasing and Λ is decreasing in the model. The same model may be constructed if we consider $-\Lambda_f$ instead of Λ_f and positive G_f. Then G is decreasing and Λ is increasing in the model.

5. Variation of ρ G and Λ

From (21) and (24), we get

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{R}}{R}(1+\beta) \qquad \text{... (33)}$$

then from (22), (26) and (33), we get

$$\frac{\dot{G}}{G} = \frac{3(1+\beta)}{(1+\alpha)}\frac{\dot{R}}{R} \qquad \text{... (34)}$$

$$\frac{\dot{\Lambda}}{\Lambda} = \frac{-3\alpha(1+\beta)}{(1+\alpha)}\frac{\dot{R}}{R} \qquad \text{... (35)}$$

Integrating eqs. (33), (34) and (35) under the conditions

$$t = t_0,\ R = R_0,\ \rho = \rho_0,\ G = G_0 \text{ and } \Lambda = \Lambda_0 \qquad \text{... (36)}$$

we get

$$\rho = \rho_0\left(\frac{R}{R_0}\right)^{-3(1+\beta)} \qquad \text{... (37)}$$

$$G = G_0\left(\frac{R}{R_0}\right)^{\frac{3(1+\beta)}{1+\alpha}} \qquad \text{... (38)}$$

and

$$\Lambda = \Lambda_0\left(\frac{R}{R_0}\right)^{\frac{-3\alpha(1+\beta)}{1+\alpha}} \qquad \text{... (39)}$$

when $1 + \alpha \neq 0$. ... (40)

The case $\alpha = -1$ has been discussed in the section 6. During the evolution of the universe we propose that the density decreases from ρ_i to ρ_0 and G decreases from G_i to G as the universe expands from R_i to R_0, that is, $\dot{\rho} < 0$ and $\dot{G} < 0$ during the expansion of the universe. Eq. (37) and (38) then implies that

$$1+\beta>0 \text{ and } 1+\alpha<0 \qquad \text{... (41)}$$

Since G and ρ are positive during the expansion of the universe and $\alpha<-1$ according to (41) hence from (26) it is clear that $\Lambda < 0$ during the expansion of the universe. Also, according to eq. (22) for $\rho>0$ and $G < 0$ we must have $\dot{\Lambda}>0$ and hence, during the expansion of the universe Λ should be negative and increasing function. On the other hand if one assumes that Λ and ρ are positive and $\dot{\Lambda}<0$, then $\dot{G}>0$ and $G < 0$ during the expansion of the universe, which is unphysical.

6. The Constants α and β

From eq. (16), we have

$$\frac{\dot{R}^2}{R^2}=8\pi G\rho=\Lambda c^2-\frac{3kc^2}{R^2}. \qquad \text{... (42)}$$

According to the initial conditions (32) at $t = 0$, the numerical value of $\frac{kc^2}{R^2}\,(k\neq 0)$ is very small (order of 10^{-6}) as compared to the numerical values of $8\pi G\rho$ and Λc^2 (both are of the order of 10^{48}). Thus, for a non-singular model of the universe satisfying the initial conditions (32) we have a condition

$$8\pi G_i\rho_i+\Lambda_i c^2=0 \qquad \text{... (43)}$$

which is same as proposed by Sivaram *et al.*[8] Eq. (26), which is our proposition for the dependence of $G\rho$ on Λ then implies

$$8\pi G_i\rho_i+\alpha\Lambda_i c^2=0 \qquad \text{... (44)}$$

at $t = 0$, comparing (43) and (44) we have

$$\alpha=-1. \qquad \text{... (45)}$$

For $\alpha=-1$, eqs. (22) and (26) together imply $G\dot{\rho}=0$ for $G\neq 0$ this gives $\dot{\rho}=0$, which on substitution in (33) gives $\dot{R}=0$ and hence we have a static model of the universe, which is unphysical.

Thus, we see that the gravitational theory under consideration (eqs. (1) and (2)) with additional requirement (26) do not permit to construct a non-singullar cosmological model of the universe, satisfying the initial conditions (32).

By choosing suitable values for the constant α and β one may have the variation of ρ, G and Λ according to eqs. (37), (38) and (39) and then from eq. (42) one may get the corresponding dependence of R on t. In this way, a number of cosmological models may be constructed and the validity of these models may be tested on the theoretical and observational grounds. Instead of approaching in this way, we propose to first determine the constants α and β by considering two variables out of the three, variables expressed in (37), (38) and (39) at a time and the corresponding numerical values at $t = 0$, and $t=t_0$. In this way, we have three types cosmological models of the universe discussed in the next section.

7. COSMOLOGICAL MODELS

From (16) and (26), we get

$$\dot{R} = c\sqrt{\left(\frac{1+\alpha}{3}\right)\Lambda R^2 - k} \qquad \text{... (46)}$$

and then putting the values of Λ from (39) in (46), we get

$$\dot{R} = c\sqrt{AR^m - k} \qquad \text{... (47)}$$

where

$$A = \frac{(1+\alpha)\,\Lambda_0}{3} R_0^{3\alpha(1+\beta)/(1+\alpha)}$$

and

$$m = \frac{2-\alpha(1+3\beta)}{1+\alpha} \qquad \text{... (48)}$$

Also, from (23), (24) and (26), we get

$$\ddot{R} = \left[1 - \frac{\alpha(1+3\beta)}{2}\right]\Lambda c^2 R \qquad \text{... (49)}$$

Eqs. (47) and (48) together will give the evolution of the cosmological model of the universe.

As discussed in the section 6, we have the following three types of the cosmological models depending on the numerical values of the physical quantities considered for the determination of the constants α and β as follows :

Type I

Let us consider eqs. (37) and (38) and the corresponding numerical values as follows

$$\left.\begin{array}{ll} \text{at} & R = R_i = 10^{13} cm,\ \rho = \rho_i = 10^{17}\ g/cm^3 \\ & G = G_i = G_f = 6.6 \times 10^{30} \text{ c.g.s unit} \\ \text{and} & R = R_0 = 10^{28}\ cm \\ & \rho = \rho_0 = 3 \times 10^{-31}\ g/cm^3 \\ & G = G_0 = 6.6 \times 10^{-8} \text{ c.g.s unit} \end{array}\right| \qquad \text{... (50)}$$

The numerical values given in (50) when substituted in (36) and (37) give

$$1+\beta = \frac{16}{15},\quad \frac{1+\beta}{1+\alpha} = -\frac{38}{45} \qquad \text{... (51)}$$

and hence

$$\alpha = -\frac{43}{19},\quad \beta = \frac{1}{15} \qquad \text{... (52)}$$

The equation (26) then gives

$$\Lambda_i = \frac{8\pi G_i \rho_i}{\alpha c^2} = -8 \times 10^{28}\ \text{cm}^{-2}$$

and

$$\Lambda_0 = \frac{8\pi G_0 \rho_0}{\alpha c^2} = -2 \times 10^{-57}\ \text{cm}^{-2} \qquad \text{... (53)}$$

Therefore, in the cosmological model Λ is negative and increasing from $-8 \times 10^{28}\ \text{cm}^{-2}$ to $-2 \times 10^{-57} \text{cm}^{-2}$ as R increases from 10^{13} cm to 10^{28} cm.

For $\Lambda < 0$, $\alpha < 0$ and $\beta > 0$, eq. (49) implies that $\ddot{R} < 0$, i.e., the model is deaccelerating. Substituting the values of α, β from (52), R_0 from (50) and Λ_0 from (53) in (48), we get

$$A = 8 \times 10^{103}\ \text{c.g.s units and } m = -\frac{56}{15} \qquad \text{... (54)}$$

and then eq. (47) gives

$$c \,.\, dt = \frac{R^{28/15}\, dR}{\sqrt{A - k\, R^{56/15}}}. \qquad \text{... (55)}$$

For $k = 1$, eq. (55) ives $\dot{R} = 0$ when $R = (A)^{15/36} = 10^{28}$ cm $= R = 0$ according to the numerical values of A as given in (54). Therefore, the model after expanding from R_i to R_0 will start contracting because $\ddot{R} < 0$. Also $\dot{R} = 0$ at $R = R_0$ implies that $H = \frac{\dot{R}}{R} = 0$ at present epoch which is not true as the observation shows that $H_0 \neq 0^{24}$. For $k = 0$.(55) is integrable under the condition $R = 0$, $t = 0$ as

$$ct = \frac{15}{43\sqrt{A}} R^{43/15} \qquad \text{... (56)}$$

This is a singular model. Eqs. (54) and (56) give

$$t_i = \frac{15}{43\sqrt{A}} \frac{R_i^{43/15}}{c} = 4 \times 10^{-26}\ \text{sec.}$$

and

$$t_0 = \frac{15}{43\sqrt{A}} \frac{R_0^{43/15}}{c} = 4 \times 10^{17}\ \text{sec.} \qquad \text{... (57)}$$

For $k = -1$, exact solution of (55) is not possible and hence, we first approximate it as follows :

For $R^{56/15} < A$ or $A^{15/56} > R$,

$$c\, dt = \frac{R^{28/15}}{\sqrt{A}} \left[1 + \frac{R^{56/15}}{A} \right]^{-1/2} dR$$

$$= \frac{R^{28/15}}{\sqrt{A}}\left[1 - \frac{1}{2A}R^{56/15} + \ldots\ldots\right]dR \qquad \ldots (58)$$

and for $A < R^{56/15}$ or $A^{15/56} < R$

$$c\ dt = \left[1 + \frac{A}{R^{56/15}}\right]^{-1/2} dR$$

$$= \left[1 - \frac{A}{2.R^{56/15}} + \ldots\ldots\ldots\right]dR. \qquad \ldots (59)$$

According to the numerical value of A given in the eq. (54), $R < A^{15/56}$ implies

$$R < 10^{28} \text{ cm} = R_0$$

and $R > A^{15/56} \Rightarrow R > 10^{28} \text{ cm} = R_0$.

Therefore, the dependence of R on t during the expansion for R_i to R_0 is governed by (58) and the expansion beyond R_0 is governed by (59). Therefore, integration (58) under the condition $R = 0$, $t = 0$, we get

$$ct = \frac{15}{43\sqrt{A}}R^{43/15} - \frac{15}{198\,A^{3/2}}R^{99/15} + \ldots\ldots,\ R < R_0 \qquad \ldots (60)$$

and integrating (59), under the condition $R = R_0$, $t = t_0$, we get

$$c(t - t_0) = (R - R_0) + \frac{15A}{82}\left(\frac{1}{R^{41/15}} - \frac{1}{R_0^{41/15}}\right) + \ldots. + \ldots.\ R \geq R_0 \qquad \ldots (61)$$

Here again, the model is singular and for the value of A given in (54) the first term on the right hand side of (60) is dominating, therefore the values of t_i and t_0 obtained from (60) are approximately same as given in (56).

Type II

Let us consider eqs. (36) and (39) and the corresponding the numerical values as follows :

$$R = R_i = 10^{13} \text{ cm};\ \rho = \rho_i = 10^{17} \text{ g/cm}^3,\ \Lambda = \Lambda_i = \Lambda_f = -10^{28} \text{ cm}^{-2}$$

and

$$R = R_0 = 10^{28};\ \rho = \rho_0 = 3 \times 10^{-31} \text{ g/cm}^3,\ \Lambda = \Lambda_0 = \Lambda_f = -10^{57} \text{ cm}^{-2}. \qquad \ldots. (62)$$

The numerical values given in (62) when substituted in (36) and (39) give

$$1 + \beta = \frac{16}{15},\ \frac{\alpha(1+\beta)}{1+\alpha} = \frac{17}{9} \qquad \ldots (63)$$

and hence, $\beta = \frac{1}{15}$ and $\alpha = -\frac{85}{37}$. $\qquad \ldots. (64)$

Eq. (26) then gives

$$\left.\begin{aligned} G_i &= \frac{\alpha \Lambda_i c^2}{8\pi\rho_i} = 8 \times 10^{30} \text{ c.g.s. units} \\ \text{and} \quad G_0 &= \frac{\alpha \Lambda_0 c^2}{8\pi\rho_j} = 3 \times 10^{-7} \text{ c.g.s.units} \end{aligned}\right| \quad \text{... (65)}$$

In this case also $\dot{R} < 0$ as may be seen from the eq. (49). Therefore, the model is deaccelerating.

Substituting the value of α, β for (64) and Λ_0 and R_0 for (62) in (48), we get

$$A = 4 \times 10^{101} \text{ c.g.s units, } m = -\frac{11}{3} \quad \text{... (66)}$$

and the eq. (47) gives

$$cdt = \frac{R^{11/6}\, dR}{\sqrt{A - kR^{11/3}}} \quad \text{... (67)}$$

As discussed in type I, the model for $k = 1$ may be rejected on the observational ground. For $k = 0$ and $k = -1$, the models are constructed on the same pattern as discussed in type I. A brief discussion is as follows :

$$k = 0, \; ct = \frac{6}{17\sqrt{A}} R^{11/6} \quad \text{... (68)}$$

Therefore

$$\left.\begin{aligned} t_i &= \frac{6}{17\sqrt{A}} \frac{R_0^{17/6}}{c} = 6 \times 10^{-25} \text{ sec} \\ t_0 & \frac{6}{17\sqrt{A}} \frac{R_0^{17/6}}{c} = 6 \times 10^{17} \text{ sec} \end{aligned}\right| \quad \text{... (69)}$$

For $k = -1$,

$$ct = \frac{6}{17\sqrt{A}} R^{16/7} - \frac{1}{13 A^{3/2}} R^{13/2}, \quad R < R_0$$

and

$$c(t - t_0) = (R - R_0) - \frac{3A}{16}\left(\frac{1}{R^{8/3}} - \frac{1}{R_0^{8/3}}\right) R > R_0. \quad \text{... (70)}$$

Type III

Let us consider eqs. (38) and (39) and the corresponding numerical values as follows :

$$\left.\begin{aligned} &R = R_i = 10^{13}\ \text{cm};\quad G = G_i = G_f = 6.6\times 10^{30}\ \text{c.g.sunits} \\ &\Lambda = \Lambda_i = \Lambda_f = -\ 10^{28}\ \text{cm}^{-2} \\ \text{and}\quad &R = R_0 = 10^{28}\ \text{cm};\quad G = G_0 = 6.6\times 10^{-8}\ \text{c.g.s.unit}, \\ &\Lambda = \Lambda_0 = -\ 10^{-57}\ cm^{-2} \end{aligned}\right| \quad \dots (71)$$

The numerical values given in (71) when substituted in (38) and (39) give

$$\frac{1+\beta}{1+\alpha} = -\frac{38}{85},\quad \frac{\alpha(1+\beta)}{1+\alpha} = \frac{17}{9} \quad \dots (72)$$

and hence, $\alpha = -\dfrac{85}{38},\ \beta = \dfrac{2}{45}$ ... (73)

Eq. (26) then gives

$$\left.\begin{aligned} &\rho_i = \frac{\alpha\Lambda_i c^2}{8\pi G_i} = 10^{17}\ \text{g/cm}^3 \\ \text{and}\quad & \\ &\rho_0 = \frac{8\Lambda_0 c^2}{8\pi G_0} = 10^{-30}\ \text{g/cm}^3 \end{aligned}\right| \quad \dots (74)$$

For $\alpha < 0, \beta > 0$ and $\Lambda < 0$, eq. (49) implies that $\ddot{R} < 0$, therefore, the model is deaccelerating. Substituting the values of α, β from (73), R_0 and Λ_0 from (71) in (48), we get

$$A = 4\times 10^{101}\ \text{c.g.s. units.}\ m = -\frac{517}{141} \quad \dots (75)$$

and eq. (48) gives

$$cdt = \frac{R^{517/282}}{\sqrt{A - kR^{517/141}}} \quad \dots (76)$$

Following the same pattern as discussed in type I, we see that for k = 1, the model is unphysical and for k=0, we have

$$ct = \frac{282}{799\sqrt{A}} R^{799/282} \quad \dots (77)$$

$$\left.\begin{aligned} t_i &= \frac{282}{799\sqrt{A}} \frac{R_i^{799/282}}{c} = 10^{-25}\ \text{sec} \\ \text{and} \\ t_0 &= \frac{282}{799\sqrt{A}} \frac{R_0^{799/282}}{c} = 10^{17}\ \text{sec} \end{aligned}\right| \quad \ldots (78)$$

For $k = -1$.

$$ct = \frac{282}{799\sqrt{A}} R^{799/282} - \frac{141}{36666\, A^{3/2}} R^{1833/282} + \ldots\ldots, \; R < R_0$$

and

$$c(t - t_0) = (R - R_0) - \frac{141A}{752}\left(\frac{1}{R^{376/141}} - \frac{1}{R_0^{376/141}}\right) + \ldots, \; R > R_0 \quad \ldots (79)$$

Concluding Remarks

In the foregoing sections we have seen that the Einstein field equations with variable G and Λ such that

$$G\rho = \frac{\alpha\pi c^2}{8\pi G}$$

and the usual conservation law implies that $\alpha < -1$, for $G > 0$ and $G < 0$, $\Lambda < 0$ and $\dot{\Lambda} > 0$. Applying the initial conditions as proposed by Isham *et al.*[23] and Sinha *et al.*[24] for the construction of a non-singular cosmological models, we have seen that there are three types of singular cosmological models depending on the determination of the constants α and β. On the observational ground a closed model is not possible. The numerical values of t_1 obtained {eqs. (57), (69) and (78)} in the three types of models are less than the time $\frac{h}{mc^2} = 10^{-23}$ sec. (m is the Hadron mass) before which the universe was in quantum era[25].

It is because of $t_i \neq 0$ and $\dot{R} \neq 0$ at $t = t_i$ the models discussed here are singular.

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