Indian J. pure appl. Math., **31**(2) : 177-183, February 2000

ON THE ABSOLUTE CESÀRO SUMMABILITY OF A SERIES RELATED TO WALSH-FOURIER SERIES

A. K. SAHOO

Department of Mathematics, Govt. Kolasib College, (Post Box No. 20) Kolasib 796081, Mizoram

(*Received* 30 *December* 1997; *after Revision* 9 *February* 1999; *Accepted* 17 *May* 1999)

The object of the present paper is to study the absolute Cesàro summability of the factored Walsh-Fourier series with the factor of the type $n^{\alpha}, 0 < \alpha < 1$.

Key Words : Absolute Cesàaro Summability; Walsh-Fourier Series; Dyadic Derivative; Strongly Differentiable

1. INTRODUCTION

1.1. Unlike trigonometric Fourier series, little attention was paid to Walsh-Fourier series. The concept of Walsh Functions and Walsh-Fourier series was given by J. L. Walsh in 1923. Kaczmarz, Steinhaus and Paley studied some aspects of Walsh system between 1929 and 1931. Fine[2] studied the (C, α) summability, $\alpha > 0$ of Walsh-Fourier series. Mohanty[3] studied the absolute Cesàro summability of the factored Fourier series with the factor of type n^{α} $(0 < \alpha < 1)$. The object of the present paper is to study the analogous result of Mohanty's result.

1.2 : The Redemacher functions are defined by

$$r_0(x) = \begin{cases} 1, \text{ if } x \in \left[0, \frac{1}{2}\right) \\ -1, \text{ if } x \in \left[\frac{1}{2}, 1\right) \end{cases}$$

$$r_0(x+1) = r_0(x), \; r_n(x) = r_0(2^n x), \; (n = 1, 2, 3, \ldots).$$

Walsh functions are then given by

$$w_0(x) = 1$$

and $w_n(x) = r_{n_1}(x)\, r_{n_2}(x) \ldots r_{n_k}(x)$

for $n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_k}$, where the integers n_i are uniquely determined with $n_{i+1} < n_i$. Any $x \in [0, 1]$ can be written in the form

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \qquad \ldots (1)$$

where each $x_k = 0$ or 1. If x is not a dyadic rational it can be uniquely expressed in the form (1). We call it dyadic expansion of x. For dyadic rationals there are two expressions of this form, one which terminates with 0's and other which terminates with 1's. By dyadic expansion of a dyadic rational we shall mean the one which terminates with 0's.

1.3 : Let $f(x)$ be a periodic function with period 1 and Lebesgue integrable over (0, 1). Then the Walsh-Fourier series of f is

$$f(t) \sim \sum_{k=0}^{\infty} c_k w_k(t) \equiv \sum_{k=0}^{\infty} A_k(t),$$

where $c_k = \int_0^1 f(u)\, w_k(u)\, du.$

We have

$$w_k(x)\, w_k(y) = w_k(x \dot{+} y),$$

where $$x \dot{+} y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)},$$

if $$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)} \text{ and } y = \sum_{k=0}^{\infty} y_k 2^{-(k+1)}$$

are dyadic expansions of x and y respectively.

So by a result due to Fine[1]

$$A_k(x) = c_k w_k(x)$$

$$= \int_0^1 f(t)\, w_k(t)\, dt\, w_k(x)$$

$$= \int_0^1 f(t)\, w_k(x \dot{+} t)\, dt$$

$$= \int_0^1 f(x \dot{+} t)\, w_k(t)\, dt.$$

1.4 : Let L^0 represent the collection of almost every-where finite Lebesgue measurable functions from [0, 1] to $[-\infty, \infty]$. For $0 < p < \infty$, let L^p represent the collection of functions $f \in L^0$ for which

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} \text{ is finite.}$$

In this paper X represents the Banach space of functions in L^1 such that

$$\tau_y f \in x, f * g \in x$$

$$\| \tau_y f \| = \|f\|$$

$$\|f\|_1 \le \|f\|$$

and $$\|f * g\| \le \|f\| \, \|g\|_1.$$

for each $f \in X$, $g \in L^1$ and $y \in [0, 1]$,

where τ and $*$ are dyadic translation and dyadic convolution on the unit interval.

DYADIC DERIVATIVE

For each function f defined on [0, 1) and non-negative integer n, let

$$d_n f(x) = \sum_{j=0}^{n} 2^{j-1} \{f(x) - f(x \dot{+} 2^{-j-1})\} \text{ for } x \in [0, 1).$$

Then f is said to be dyadically differentiable at x if $f^{[1]}(x) = \lim_{n \to \infty} d_n f(x)$ exists and finite and $f^{[1]}(x)$ is called dyadic derivative of f at x. A function f is called strongly differentiable in X if there exists a function $g \in X$ such that

$$d_n f \to g \text{ in the norm of } X \text{ as } n \to \infty.$$

2. PURPOSE OF THE PRESENT WORK

The object of this paper is to prove the following theorem :

Theorem — *Let* $\phi(t) = f(x \dot{+} t) - S$, *where S is a function of x. If* $\phi(t)$ *is strongly differentiable in X and* $\int_0^1 \frac{|\phi^{[1]}(t)|}{t^\alpha} dt < \infty, 0 < \alpha < 1$, *then the series* $\sum_{n=1}^{\infty} n^\alpha A_n(x)$ *is summable* $|c, \beta|, \beta > \alpha$.

3. Notations, Lemmas and Estimations

3.1. We need the following notations

$$T=\left[\frac{1}{t}\right]$$

$$A_n^{\beta}=\binom{n+\beta}{n}=\frac{(\beta+1)(\beta+2)\ldots(\beta+n)}{n!}\cong\frac{n^{\beta}}{(\beta+1)}(\beta\neq-1,-2,\ldots),$$

$$E^{\beta}(n,t)=\frac{1}{A_n^{\beta}}\sum_{k=0}^{n}A_{n-k}^{\beta-1}\ (k+1)^{\alpha}W_{k+1}(t)$$

and $$D_n(t)=\sum_{k=0}^{n-1}w_k(t)\ .$$

3.2. We need the following lemmas for the proof of our theorem,

Lemma 1 — ([4]) page 35, Theorem 10)

For $0<t<1$

$$|D_n(t)|<\min\left(n,\frac{2}{t}\right)$$

Lemma 2 — If f is strongly differentiable in X,

$$f\sim\sum_{n=0}^{\infty}a_nw_n \text{ and } f^{[1]}\sim\sum_{n=0}^{\infty}b_nw_n,$$

then $$b_n=n\,a_n.$$

The proof of this lemma is similar to that of Theorem 13 ([4], page 42).

3.3 : We need the following estimations for the proof of our theorem.

For $0<\alpha<\beta<1$

(a) $E^{\beta}(n,t)=0(n^{\alpha})$

(b) $E^{\beta}(n,t)=0\,(n^{\alpha-\beta}t^{-\beta})$ for $n>T$.

Proof of (A) :

$$|E^{\beta}(n,t)|\leq\frac{1}{A_n^{\beta}}\sum_{k=0}^{n}A_{n-k}^{\beta-1}\ (k+1)^{\alpha}$$

$$=\ 0(n^{\alpha})..$$

PROOF OF (B) :

$$E^{\beta}(n,t)=\frac{1}{A_n^{\beta}}\sum_{k=0}^{n} A_{n-k}^{\beta-1}\ (k+1)^{\alpha}\ W_{k+1}(t)$$

$$=\frac{1}{A_n^{\beta}}\sum_{k=0}^{n-T} A_{n-k}^{\beta-1}\ (k+1)^{\alpha} w_{k+1}(t)+\frac{1}{A_n^{\beta}}\sum_{k=n-T+1}^{n} A_{n-k}^{\beta-1}\ (k+1)^{\alpha} w_{k+1}(t)$$

Since $A_{n-k}^{\beta-1}$ increases with k for $0<\beta<1$, we have

$$\left|\frac{1}{A_n^{\beta}}\sum_{k=0}^{n-T} A_{n-k}^{\beta-1}\ (k+1)^{\alpha} w_{k+1}(t)\right|\le\frac{A_T^{\beta-1}}{A_n^{\beta}}\max_{0<L,L'<n-T}\left|\sum_{L}^{L'}(k+1)^{\alpha} w_{k+1}(t)\right|$$

$$=\frac{A_T^{\beta-1}}{A_n^{\beta}}0\left(\frac{n^{\alpha}}{t}\right)\text{ by Lemma 1}$$

$$=0\left(\frac{T^{\beta-1}}{n^{\beta}}\cdot\frac{n^{\alpha}}{t}\right)$$

$$=0(n^{\alpha-\beta}t^{-\beta});$$

and

$$\left|\frac{1}{A_n^{\beta}}\sum_{k=n-T+1}^{n}(k+1)^{\alpha}\ A_{n-k}^{\beta-1}\ w_{k+1}(t)\right|\le\frac{1}{A_n^{\beta}}\sum_{k=n-T+1}^{n} A_{n-k}^{\beta-1}\ (k+1)^{\alpha}$$

$$\le\frac{(n+1)^{\alpha}}{A_n^{\beta}}\sum_{k=n-T}^{n} A_{n-k}^{\beta-1}$$

$$=\frac{(n+1)^{\alpha}}{A_n^{\beta}}\cdot A_T^{\beta}$$

$$=0\,(n^{\alpha-\beta}t^{-\beta}).$$

This completes the proof of (*b*).

4. PROOF OF THE THEOREM

Without loss of generality we can assume that $0<\alpha<\beta<1$. It is enough to show that

$$\sum_{n=1}^{\infty} \frac{\left|\xi_n^{\beta}\right|}{n} < \infty,$$

where ξ_n^{β} is the n-th Cesàro mean of order β of the sequence

$$\{(\nu + 1)^{\alpha + 1} A_{\nu + 1}^{n} (x)\}.$$

By Lemma 2,

$$(k+1) A_{k+1} (x) = (k+1) \int_0^1 \phi(t) w_{k+1} (t)\, dt$$

$$= \int_0^1 \phi^{[1]} (t) w_{k+1} (t)\, dt$$

So,

$$\xi_n^{\beta} = \frac{1}{A_n^{\beta}} \sum_{k=0}^{n} A_{n-k}^{\beta - 1} (k+1)^{\alpha + 1} A_{k+1} (x)$$

$$= \frac{1}{A_n^{\beta}} \sum_{k=0}^{n} A_{n-k}^{\beta - 1} (k+1)^{\alpha} \int_0^1 \phi^{[1]} (t) w_{k+1} (t)\, dt$$

$$= \int_0^1 \phi^{[1]} (t) E^{\beta} (n, t)\, dt$$

Since $\int_0^1 \frac{|\phi^{[1]} (t)|}{t^{\alpha}} dt < \infty$, it is sufficient to show that it is uniform in $0 < t < 1$.

$$\sum_{n=1}^{\infty} \frac{|E^{\beta} (n, t)|}{n} = 0\left(\frac{1}{t^{\alpha}}\right).$$

Now

$$\sum_{n=1}^{\infty} \frac{|E^{\beta} (n, t)|}{n} = \sum_{1}^{T} + \sum_{T+1}^{\infty} \quad \text{, (say).} \qquad \ldots (2)$$

Using estimation (a)

$$\sum_{1}^{T} = 0\left(\sum_{n=1}^{T} n^{\alpha - 1}\right)$$

$$= 0(T^{\alpha})$$

$$= 0\left(\frac{1}{t^{\alpha}}\right). \qquad \dots (3)$$

Using estimation (b)

$$\sum_{T+1}^{\infty} = 0\left(\sum_{n=T+1}^{\infty} \frac{n^{\alpha-\beta}t^{-\beta}}{n}\right)$$

$$= 0\left(\frac{1}{t^{\beta}} \sum_{n=T+1}^{\infty} \frac{1}{n^{\beta-\alpha+1}}\right)$$

$$= \left(\frac{1}{t^{\alpha}}\right). \qquad \dots (4)$$

Combining (2), (3) and (4) we have

$$\sum_{n=1}^{\infty} \frac{|E^{\beta}(n,t)|}{n} = \left(\frac{1}{t^{\alpha}}\right).$$

This completes the proof of the theorem.

I am grateful to Dr B.K. Ray for his valuable suggestions during the preparation of this paper.

The valuable suggestion and criticisms of the referee which led to the improvement of this paper, is also acknowledged.

REFERENCES

1. N. J. Fine, *Trans. Amer. math. Soc.* **65** (1949), 372-414.
2. *Proc. Natl. Acad. Sci. USA.* **41** (1955), 588-91.
3. R. Mohanty, *J. of London math. Soc.* **25** (1950), 63-67.
4. F. Schipp, W. R. Wade and P. Simon *'Walsh series, An Introduction to Dyadic Harmonic Analysis' Akademica, Kiado, Budapest*, 1990.