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EXISTENCE AND APPROXIMATION RESULTS FOR ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

B. K. SHARMA* AND D. R. SAHU

School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur 492010

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The purpose of the present paper is to provide existence and convergence results for fixed points of asymptotically pseudocontractive mappings in Banach spaces. One of the convergence results contained in this paper improves the result of Schu[11].

Key Words : Fixed Point; Asymptotically Pseudocontractive; Fixed Point Iteration Process; Uniformly Convex Banach Space

1. INTRODUCTION

In a recent paper, Schu[11] introduced a class of asymptotically pseudocontractive mappings in Hilbert space and proved strong convergence theorem for said class of mappings using modified Ishikawa iteration scheme as below :

Theorem 1.1 — *Let $(E, (\cdot,\cdot))$ be a Hilbert space; $\phi \neq A \subset E$ closed bounded and convex; $L > 0$; $T: A \to A$ completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\}$ in $[1, \infty)$; $q_n = 2k_n - 1$ for all $n \in I\!N$; $\sum_{n=1}^{\infty} (q_n^2 - 1) < \infty$; $\{\alpha_n\}$, $\{\beta_n\}$ in $[0, 1]$; $\varepsilon \leq \alpha_n \leq \beta_n \leq b$ for all $n \in I\!N$, some $\varepsilon > 0$ and some $b \in (0, L^{-2}[(1 + L^2)^{1/2} - 1])$; $x_1 \in A$; for all $n \in I\!N$, define*

$$x_{n+1} = \alpha_n T^n(z_n) + (1 - \alpha_n)x_n \text{ and } z_n = \beta_n T^n(x_n) + (1 - \beta_n)x_n. \quad \dots (1.1)$$

Then $\{x_n\}$ converges strongly to some fixed point of T.

Clearly T is completely continuous. This is a strong condition which insures not only the existence of fixed point of T but also the strong convergence of sequence $\{x_n\}$ defined by (1.1).

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This is the reason why Schu[11] retained this condition in his result. Now, we claim to drop the compactness from this condition. With this objective, we first give the existence of the fixed point of asymptotically pseudocontractive mappings in Banach spaces. Then we prove the weak and strong convergence result via modified Ishikawa iteration scheme for asymptotically pseudocontractive mappings in Banach spaces having property $(U, \lambda, m = 1, m)$, $\lambda \in I\!R$ $m \in I\!N$. The Banach spaces with the property $(U, \lambda, m+1, m)$, $\lambda \in I\!R$, include the L_p (or l_p) spaces, $p \geq 2$.

2. Preliminaries and Lemmas

A normed space $(E, \|\cdot\|)$ is called uniformly convex if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ it follows that $\|x + y\| \leq 2(1 - \delta)$.

$(E, \|\cdot\|)$ is said to satisfy Opial's condition[8] if for each sequence $\{x_n\}$ weakly converging to a point $x \in E$ and for all $y \in E$, it follows from $y \neq x$ that

$$\liminf \|x_n - x\| < \liminf \|x_n - y\|.$$

Let $(E, \|\cdot\|)$ be a real Banach space. For a given gauge function μ, this means for a mapping $\mu : I\!R \to I\!R$ which is continuous and strictly increasing with $\mu(0) = 0$ and $\lim_{x \to \infty} \mu(x) = \infty$, the related set-valued duality mapping $J_E^{\mu} : E \to 2^{E^*}$ is given by

$$J_E^{\mu}(x) = \{u \in X^* : u(x) = \|u\| \|x\| \text{ and } \|u\| = \mu(\|x\|)\} \text{ for all } x \in E.$$

A mapping $J : E \to E^*$ is called duality mapping with respect to μ if $J(x) \in J_E^{\mu}(x)$ for all $x \in E$. Such a mapping J is said to be weakly sequentially continuous if for all $\{x_n\}$ in E and all $x \in E$, it follows from $x_n \xrightarrow{w} x$ that $J(x_n) \xrightarrow{w^*} J(x)$ (as usual $\xrightarrow{w}$ and $\xrightarrow{w^*}$ stand for weak and weak* convergence, respectively, while strong convergence of a sequence $\{z_n\}$ to a point z in E is indicated by $\lim z_n = z$). It is well known that J_E^{μ} is single-valued if and only if $(E, \|\cdot\|)$ is smooth (see, for example, [4, page 22].

In all our proof we assume, without loss of generality, that J_E^{μ} normalized i.e., $\mu = id$ (the identity mapping).

Following [12] and [14], let $(E, \|\cdot\|)$ be a normed space; $p, \alpha \geq 0 : \beta, b \in I\!R$.

(*i*) ([7]), [12]), $(E, \|\cdot\|)$ is said to satisfy Hanner inequality with constant p iff

$$\|x + y\|^p + \|x - y\|^p \leq (\|x\| + \|y\|)^p + |\|x\| - \|y\||^p \text{ for all } x, y \in E.$$

(*ii*) ([12], [14]), $(E, \|\cdot\|)$ has property (U, b, α, β) iff

$$\|x + y\|^{\alpha} + b\|x - y\|^{\alpha} - 2^{\beta}(\|x\|^{\alpha} + \|y\|^{\alpha}) \geq 0 \text{ for all } x, y \in E.$$

(*iii*) ([12]) $(E, \|\cdot\|)$ is an upper weak parallelogram space with constant b ($(E, \|\cdot\|) \in UWP(b)$ for short) iff

$$\| x+y \|^2 + b \|x-y\|^2 \geq 2 (\| x \|^2 + \| y \|^2) \text{ for all } x, y \in E.$$

From [7], (Theorem 1), L_p (and l_p), $p \in [2, \infty)$ satisfy Hanner inequality with constant p, and from [14], (Theorem 4), every normed space satisfying Hanner inequality for some $p \in [2, \infty)$ also possesses property $(U, p-1, 2, 1)$. Hence, L_p and l_p serve as examples of Banach spaces with property $(U, \alpha, m+1, m)$.

Let us now recall some definitions and introduce a new property which we will call "asymptotically hemicontractive" :

Definition 2.1 — Let $(E, \|\cdot\|)$ be a normed space; $\phi \neq D \subset E; T : D \to D$; $\{k_n\} \subset [1, \infty)$; $L > 0$ and $F(T) = \{x \in D : T(x) = x\}^-$. Then

(*i*) T is said to be asymptotically pseudocontractive with sequence $\{k_n\}$ if $\lim (k_n) = 1$ and for all $n \in I\!N$ and all $x, y \in D$ there is $u \in J_E(x-y)$ such that $u(T^n(x) - T^n(y)) \leq k_n \| x-y \|^2$, where J_E is the normalized duality mapping.

(*ii*) T is said to be asymptotically hemicontractive with sequence $\{k_n\}$, if $\lim (k_n) = 1$, for $y \in F(T)$ and for all $x \in D$ and $n \in I\!N$ there is $u \in J_E(x-y)$ such that $u(T^n(x) - y) \leq k_n \| x-y \|^2$.

(*iii*) T is said to be uniformly L-Lipschitzian if

$$\| T^n(x) - T^n(y) \| \leq L \| x-y \| \text{ for all } n \in I\!N \text{ and all } x, y \in D.$$

(*iv*) T is called uniformly asymptotically regular if for each $\varepsilon > 0$ there is $n_0 \in I\!N$ such that $\| T^n(x) - T^{n+1}(x) \| \leq \varepsilon$ for all $n \geq n_0$ and all $x \in D$.

The notion of asymptotically hemicontractiveness is more general than that of asymptotically pseudocontractiveness. Indeed, an asymptotically pseudocontractive mapping with at least one fixed point is asymptotically hemicontractive, but there exist asymptotically hemicontractive mappings which are not asymptotically pseudocontractive.

The following example due to Rhoades[9] shows that the class of asymptotically nonexpansive mappings is a proper subclass of the class of asymptotically pseudocontractive mappings. For $x \in [0, 1]$ define $T(x) = (1 - x^{2/3})^{3/2}, x \in [0, 1]$. Then T is not Lipschitzian (see [9]) and so it can't be asymptotically nonexpansive. But since $T \cdot T = id$ and T is monotonically decreasing, it follows that

$$(x-y)(T^n x - T^n y) = | x-y |^2 \text{ for all } n \in 2 I\!N$$

and

$$(x-y)(T^n x - T^n y) = (x-y)(Tx - Ty) \leq 0$$

$$\leq | x-y |^2 \text{ for all } n \in 2I\!N - 1.$$

Hence, T is asymptotically pseudocontractive with constant sequence $(1)_{n \in \mathbb{N}}$.

Let $(E, \|\cdot\|)$ be nromed space, $\phi \neq D \subseteq E$ and $\{x_n\}$ be a bounded sequence. For each $y \in E$ set $r(\{x_n\}, y) = \limsup_n \|x_n - y\|$. The *asymptotic radius* of $\{x_n\}$ with respect to D is defined by

$$R(\{x_n\}, D) = \inf_{y \in D} r(\{x_n\}, y),$$

and the asymptotic center of $\{x_n\}$ with respect to D is given by

$$AC(\{x_n\}, D) = \{y \in D : r(\{x_n\}, y) = R(\{x_n\}, D\}.$$

Lemma 2.1[2] — Let $(E, \|\cdot\|)$ be a uniformly convex Banach space; $\phi \neq D \subseteq E$ be closed and convex; $\{x_n\}$ be a bounded sequence in D and $z \in D$ with $AC(\{x_n\}, D) = \{z\}^{-}$. Suppose $\{y_n\}$ is a sequence in D such that

$$\lim_{n \to \infty} r(\{x_i\}, y_n) = R(\{x_i\}, D).$$

Then $\lim y_n = z$.

Lemma 2.2 — Let $(E, \|\cdot\|)$ be a Banach space possessing a weakly sequentially continuous duality mapping and $\phi \neq D \subseteq E$. Suppose $\{x_n\}$ be a sequence in D such that $x_n \xrightarrow{W} x$ in D. Then $AC(\{x_i\}, D) = \{x\}$.

PROOF : It follows from [6, Lemma 3].

Lemma 2.3[10]: Let $(E, \|\cdot\|)$ be a normed space with property $(U, \lambda, m+1, m)$, $\lambda \in \mathbb{R}$ $m \in \mathbb{N}$. Then

$$\|x+y\|^{m+1} \leq \|x\|^{m+1} + \left(\frac{\lambda}{2^m - 1}\right)\|y\|^{m+1} + (m+1)\|x\|^{m-1} u(y) \qquad \ldots (1)$$

for all $x, y \in E$, and $u \in J_E(x)$.

Lemma 2.4: Let $(E, \|\cdot\|)$ be a normed space with property $(U, \lambda, m+1, m)$, $\lambda \in \mathbb{R}$ $m \in \mathbb{N}$ $\phi \neq D \subseteq E; T : D \to D$ and $\{k_n\} \subseteq [1, \infty)$. Then the following are equivalent :

(*a*) $u(T^n(x) - T^n(y)) \leq k_n \|x - y\|^2$ for all $n \in \mathbb{N}$ and all $x, y \in D$, where $u \in J_E(x - y)$.

(*b*) $\|T^n(x) - T^n(y)\|^{m+1} \leq [(m+1)k_n - m]\|x - y\|^{m+1} +$

$$\left(\frac{\lambda}{2^m - 1}\right)\|(id - T^n)x - (id - T^n)y\|^{m+1} \qquad \ldots (2)$$

for all $n \in \mathbb{N}$ and $x, y \in D$.

Above lemma can be shown by a simple calculation.

Finally, we investigate the following Lemma which plays a key role in this paper.

Lemma 2.5 — Let $(E, \|\cdot\|)$ be a uniformly convex Banach space possessing a weakly sequentially continuous duality mapping; $\phi \neq D \subseteq E$ be closed and convex; and $T: D \to D$ a uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\}$ satisfying the condition (*) :

$$(*) \quad \| x - T^n y \|^2 \leq u(x - T^n y) \text{ for all } x, y \in D, n \in I\!N,$$

where $u \in J_E(x-y)$. Then *id-T* is demiclosed with respect to 0.

PROOF : Suppose $\{x_n\}$ be a bounded sequence in D satisfies $\lim \| x_n - Tx_n \| = 0$. Since X is reflexive then $x_n \xrightarrow{w} x$ for some x in D. Using (*), then for $m, n \in I\!N$ and $u \in J_E(x_n - x)$, we have

$$\| x_n - T^m x \|^2 \leq u(x_n - T^m x)$$

$$= u((x_n - Tx_n) + (Tx_n - T^2 x_n) + \dots + (T^m x_n - T^m x))$$

$$\leq mL \| x_n - Tx_n \| \| x_n - x \| + k_m \| x_n - x \|^2$$

Taking limit superior as $n \to \infty$ in above inequality, we obtain for all $m \in I\!N$,

$$\limsup \| x_n - T^m(x) \| \leq \limsup \| x_n - x \|,$$

hence

$$r(x_n\}, T^m(x) \leq r(\{x_n\}, x).$$

Then, we have, as $n \to \infty$,

$$r(\{x_n\}, T^m(x)) = R(\{x_n\}, D),$$

it follows from Lemma 2.1 that

$$\lim_{m \to \infty} T^m(x) = x.$$

By continuity of T, this leads to $Tx = x$.

3. Fixed Point Theorems for Asymptotically Pseudocontractive Mappings

In this section, we first establish a fixed point theorem for an asymptotically pseudocontractive mapping defined on a bounded convex subset of a Banach space :

Theorem 3.1 — Let $(E, \|\cdot\|)$ be a Banach space; $\phi \neq D \subseteq E$ be closed and convex; $T: D \to D$ is asymptotically pseudocontractive with sequence $\{k_n\}$ and $\lambda_n \in (0, 1)$ for all $n \in I\!N$ with $\lim \lambda_n = 1$. Then *(a) for each* $n \in \mathbb{N}$ *there is exactly one* $x_n \in D$ *such that*

$$x_n = (\lambda_n / k_n)\, T^n(x_n), \text{ where } \lambda_n \in (0, 1),$$

(b) if D is bounded and T is uniformly asymptotically regular and uniformly L-Lipschitzian, it follows that $\inf \{\| x - Tx \| : x \in D\}^{-} = 0$,

PROOF : Without loss of generality we may assume that $0 \in D$.

(a) for $n \in \mathbb{N}$ define $\lambda_n = 1 - 1/(n+1)$ and $F_n := (\lambda_n / k_n) T^n$. Then, since is convex, the mapping F_n maps D into itself for each fixed $n \in \mathbb{N}$. Hence, for $x, y \in D$ there exists $u \in J_E(x - y)$ such that

$$u(F_n(x) - F_n(y)) = (\lambda_n / k_n)\, u(T^n(x) - T^n(y))$$

$$\leq \lambda_n \| x - y \|^2,$$

it follows from Corollary 1 of [3] that for each $n \in \mathbb{N}$, there exists an $x_n \in D$ such that $x_n = F_n(x_n) = (\lambda_n / k_n) T^n(x_n)$.

(b) Since D is bounded, then we have

$$\| x_n - T^n(x_n) \| \leq | 1 - (\lambda_n / k_n) | \text{ diam } (D)$$

for all $n \in \mathbb{N}$ and so

$$\lim \| (x_n) - T^n(x_n) \| = 0.$$

By the uniformly asymptotic regularity of T, we have

$$\lim \| T^n(x_n) - T^{n-1}(x_n) \| = 0.$$

Furthermore, for all $n \in \mathbb{N}$

$$\| x_n - Tx_n \| \leq \| x_n - T^n(x_n) \| + \| T^n(x_n) - Tx_n \|$$

$$\leq \| x_n - T^n(x_n) \| + L \| T^{n-1}(x_n) - x_n \|$$

$$\leq \| x_n - T^n(x_n) \| + L (\| T^{n-1}(x_n) - T^n(x_n) \| + \| T^n(x_n) - x_n \|).$$

Taking limit as $n \to \infty$, we obtain $\lim \| x_n - Tx_n \| = 0$, which establishes

$$\inf \{\| x - Tx \| : x \in D\}^{-} = 0.$$

We now prove a fixed point existence theorem for asymptotically pseudocontractive mappings.

Theorem 3.2 — *Let* $(E, \| \cdot \|)$ *be a uniformly convex Banach space possessing a weakly sequentially continuous duality mapping and* $\phi \neq D \subseteq E$ *be bounded closed and convex. Suppose*

$T: D \to D$ is uniformly asymptotically regular, uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\}$ and satisfying the condition (*). Then $F(T) \neq \phi$.

PROOF : By Theorem 3.1, for each $n \in IN$ there exists one $x_n \in D$ such that $x_n = (\lambda_n/k_n) T^n(x_n)$, where $\lambda_n \in (0, 1)$ with $\lim (\lambda_n) = 1$, since $\{x_n\}$ is bounded. Therefore, there is a constant K such that $\| x_n - T^n(x_n) \| = | 1 - (k_n/\lambda_n) | K$ for all $n \in IN$ and so $\| x_n - T^n(x_n) \| \to 0$ as $n \to \infty$. By the asymptotically regularity of T, we have

$$\| x_n - Tx_n \| \leq \| x_n - T^n(x_n) \| + \| T^n (x_n) - Tx_n \|$$

$$\leq \| x_n - T^n(x_n) + L \| T^{n-1} (x_n) - x_n \|$$

$$\leq \| x_n - T^n(x_n) \| + L (\| T^{n-1} (x_n) - T^n (x_n) \| + \| T^n(x_n) - x_n \|)$$

$$\to 0 \text{ as } n \to \infty.$$

Since $(E, \| \cdot \|)$ is reflexive and $\{x_n\}$ is bounded, there exists $z \in E$ and a subsequence $\{x_{\psi_n}\}$ of $\{x_n\}$ such that $x_{\psi_n} \xrightarrow{w} z$ (Pettis' theorem).

Furthermore, since $x_{\psi_n} - Tx_{\psi_n} \to 0$ and by Lemma 2.5 $(id - T)$ is demiclosed at zero, it follows that $z = Tz$. This completes the proof.

4. Weak and Strong Convergence of Ishikawa Iteration Scheme

In this section, we turn to consider approximating fixed points for asymptotically hemicontractive and asymptotically pseudocontractive mappings via the modified Ishikawa iteration scheme. First, we prove weak convergence theorem for asymptotically hemicontractive and asymptotically pseudocontractive mappings in a uniformly convex Banach space which satisfying Opial's condition via the modified Ishikawa iteration scheme.

Theorem 4.1 — *Let $(E, \| \cdot \|)$ be a Banach space with property $(U, \lambda, m+1, m)$, $\lambda \in IR$, $m \in IN$ satisfying Opial's condition; $\phi \neq D \subseteq E$ be closed, convex and bounded; $T: D \to D$ is uniformly L-Lipschitzian for some $L > 0$ and asymptotically hemicontractive with sequence $\{k_n\}$ and id-T is demiclosed with respect to zero. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the following conditions :*

(*i*) $0 < a \leq \alpha_n \leq \alpha < 1$ and $0 < b \leq \beta_n \leq \beta < 1$ for all $n \in IN$,

(*ii*) $\sum_{n=1}^{\infty} (v_n c - m) < \infty$, where $v_n = (m+1) k_n - m$ and $c = \left(\dfrac{\lambda}{2^m - 1} \right)$,

(*iii*) $(1 - 2\beta^m c - \beta^{m+1} L^{m+1} c) c + 1 - \beta^m c - c^2 > 0$ and $1 - \alpha^m c - (1 - mb) c^2 > 0$.

Pick $x_1 \in D$ and define $x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n(y_n)$ and $y_n = (1 - \beta_n) x_n + \beta_n T^n(x_n)$ for all $n \in IN$.

Then (*a*) $\lim \| x_n - T^n(x_n) \| = 0$,

(*b*) $\{x_n\}$ converges weakly to an element of $F(T)$.

PROOF : For any x, y, z in a real Banach space with property $(U, \lambda, m+1, m)$, $\lambda \in \mathbb{R}$, $m \in \mathbb{N}$ and real number $t \in [0, 1]$, we have

$$\| (1-t)x + ty - z \|^{m+1} \leq (1-mt) \| x - z \|^{m+1} + tc \| y - z \|^{m+1} - t[1 - t^m c] \| x - y \|^{m+1}. \quad \text{... (3)}$$

Then for $p \in F(T)$ and each $n \in \mathbb{N}$

$$\| x_{n+1} - p \|^{m+1} = \| (1-\alpha_n)x_n + \alpha_n T^n(y_n) - p \|^{m+1}$$
$$\leq (1 - m\alpha_n) \| x_n - p \|^{m+1} + \alpha_n c \| T^n(y_n) - p \|^{m+1} - \alpha_n (1 - \alpha_n^m c) \| x_n - T^n(y_n) \|^{m+1}. \quad \text{... (4)}$$

Since T is asymptotically hemicontractive, we have

$$\| T^n(x_n) - p \|^{m+1} \leq v_n \| x_n - p \|^{m+1} + c \| x_n - T^n(x_n) \|^{m+1}. \quad \text{... (5)}$$

Similarly, we have

$$\| T^n(y_n) - p \|^{m+1} \leq v_n \| y_n - p \|^{m+1} + c \| y_n - T^n(y_n) \|^{m+1}. \quad \text{... (6)}$$

Using inequality (3), we obtain

$$\| y_n - p \|^{m+1} = \| (1-\beta_n) x_n + \beta_n T^n(x_n) - p \|^{m+1}$$
$$\leq (1 - m\beta_n) \| x_n - p \|^{m+1} + \beta_n c \| T^n(x_n) - p \|^{m+1} - \beta_n (1 - \beta_n^m c) \| x_n - T^n(x_n) \|^{m+1}.$$

Substitution of inequality (5) into the above inequality, gives

$$\| y_n - p \|^{m+1} \leq [1 + \beta_n (v_n c - m)] \| x_n - p \|^{m+1} + \beta_n [c^2 + \beta_n^m c - 1] \| x_n - T^n(x_n) \|^{m+1}. \quad \text{... (7)}$$

Substitution of (7) in (6) yields :

$$\| T^n(y_n) - p \|^{m+1} \leq v_n (1 + \beta_n (v_n c - m)) \| x_n - p \|^{m+1} + \beta_n v_n (c^2 + \beta_n^m c - 1) \| x_n - T^n(x_n) \|^{m+1} + c \| y_n - T^n(y_n) \|^{m+1}. \quad \text{... (8)}$$

Moreover,

$$\| y_n - T^n (y_n) \|^{m+1} = \| (1-\beta_n) x_n + \beta_n T^n(x_n) - T^n (y_n) \|^{m+1}$$

$$\leq (1 - m\beta_n) \| x_n - T_n (y_n) \|^{m+1} + \beta_n c \| T^n(x_n) - T^n(y_n) \|^{m+1}$$

$$- \beta_n (1 - \beta_n^m c) \| x_n - T^n (x_n) \|^{m+1}$$

$$\leq (1 - m\beta_n) \| x_n - T^n (y_n) \|^{m+1} + \beta_n c L^{m+1} \beta_n^{m+1} \| x_n - T^n (x_n) \|^{m+1}$$

$$- \beta_n (1 - \beta_n^m c) \| x_n - T^n(x_n) \|^{m+1}$$

$$= (1 - m\beta_n) \| x_n - T^n (y_n) \|^{m+1} - \beta_n (1 - \beta_n^m c - \beta_n^{m+1} L^{m+1} c)$$

$$\| x_n - T^n (x_n) \|^{m+1}. \qquad \ldots (9)$$

Substitution of inequality (9) in (8) gives :

$$\| T^n(y_n) - p \|^{m+1} \leq v_n (1 + \beta_n (v_n c - m)) \| x_n - p \|^{m+1}$$

$$- \beta_n [(1 - \beta_n^m c - \beta_n^{m+1} L^{m+1} c) c - v_n (c^2 + \beta_n^m c - 1)] \| x_n - T^n (x_n) \|^{m+1}$$

$$+ (1 - m\beta_n) c \| x_n - T^n (y_n) \|^{m+1} \qquad \ldots (10)$$

Using inequalities (4) and (10), we obtain

$$\| x_{n+1} - p \|^{m+1} \leq \| x_n - p \|^{m+1} + \alpha_n [v_n c(1 + \beta_n (v_n c - m)) - m] \| x_n - p \|^{m+1}$$

$$- \alpha_n \beta_n c [1 - \beta_n^m c - \beta_n^{m+1} L^{m+1} c) c - v_n (c^2 + \beta_n^m c - 1)] \| x_n - T^n (x_n) \|^{m+1}$$

$$- \alpha_n [1 - \alpha_n^m c - (1 - m\beta_n) c^2] \| x_n - T^n (y_n) \|^{m+1}$$

$$\leq \| x_n - p \|^{m+1} + \alpha_n (1 + \beta_n v_n c)(v_n c - m) \| x_n - p \|^{m+1}$$

$$- \alpha_n \beta_n c [(1 - \beta_n^m c - \beta_n^{m+1} L^{m+1} c) c - v_n (c^2 + \beta_n^m c - 1] \| x_n - T^n (x_n) \|^{m+1}$$

and hence

$$\alpha_n \beta_n c [(1 - \beta_n^m c - \beta_n^{m+1} L^{m+1} c) c - v_n (c^2 + \beta_n^m c - 1)] \| x_n - T^n (x_n) \|^{m+1}$$

$$\leq \| x_n - p \|^{m+1} - \| x_{n+1} - p \|^{m+1} + \alpha_n Md(v_n c - m),$$

where $M = (1 + \max_{n \in I\!N} \beta v_n c)$ and d = diam (D).

Summing from 1 to r, we have

$$0 < abc\ [(1 - 2\beta^m c - \beta^{m+1} L^{m+1} c)c + 1 - \beta^m - c - c^2] \sum_{n=1}^{r} \| x_n - T^n (x_n) \|^{m+1}$$

$$\leq \| x_1 - p \|^{m+1} + \alpha \cdot Md \sum_{n=1}^{r} (v_n c - m).$$

Since $\sum_{n=1}^{\infty} (v_n c - m)$ converges, it follows that $\lim \| x_n - T^n (x_n) \| = 0$. Now, since T is uniformly L-Lipschitzian for some $L > 0$, it follows from Lemma 1.2 of [9] that $\lim \| x_n - T (x_n) \| = 0$. Consider two subsequences $\{x_{\varphi_n}\}$ and $\{x_{\psi_n}\}$ of $\{x_n\}$ which are weakly convergent to some points x and z in D, respectively. Since $\lim \| x_n - T(x_n) \| = 0$ and *id-T* is demiclosed with respect to zero, it follows that $Tx = x$ and $Tz = z$.

Assuming that $x \neq z$ and taking into account the fact $x_{\varphi_n} \overset{w}{\to} x$ and $x_{\psi_n} \overset{w}{\to} z$, it follows from Opial's condition that

$$\lim\inf \| x_{\varphi_n} - x \| < \lim\inf \| x_{\varphi_n} - z \|,$$

and $$\lim\inf \| x_{\psi_n} - z \| < \lim\inf \| x_{\psi_n} - x \|,$$

which is a contradiction. Thus there exists exactly one cluster point p of $\{x_n\}$ in D. Therefore, all weakly convergent subsequences of $\{x_n\}$ have the same limit p. This clearly implies $\{x_n\}$ converges weakly to $p \in F(T)$, converges weakly to $p \in F(T)$, completing the proof.

***Theorem** 4.2 — Let $(E, \| \cdot \|)$ be a Banach space with property $(U, \lambda, m+1, m)$, $\lambda \in I\!R$, $m \in I\!N$ satisfying Opial's condition; $\phi \neq D \subseteq E$ be closed convex and bounded and $T : D \to D$ is uniformly L-Lipschitzian for some $L > 0$, asymptotically pseudocontractive with sequence $\{k_n\}$ and satisfying the condition (*). Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the conditions (i), (ii) and (iii).*

Pick $x_1 \in D$ and define $x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n (y_n)$ and $y_n = (1 - \beta_n) x_n + \beta_n T^n (x_n)$ for all $n \in I\!N$.

Then (*a*) $\lim \| x_n - T^n (x_n) \| = 0$,

and (*b*) $\{x_n\}$ converges weakly to an element of $F(T)$,

PROOF : The existence of fixed a point to T in D follows from [9, Remark]. Since T is asymptotically pseudocontractive and $F(T) \neq \phi$, it follows that T is asymptotically hemicontractive. The result follows from Theorem 4.1.

Next we discuss the strong convergence of the modified Ishikawa iteration scheme.

Recall that a mapping $T: D \to D$ is said to satisfy *Condition A* ([13]) if there exists a nondecreasing function $f: I\!R^+ \to I\!R^+$ with $f(0) = 0$ and $f(t) > 0$ for each $t > 0$ such that $\| x - Tx \| \geq f(d(x, F(T)))$ for all $x \in D$, where $d(x, F(T)) = \inf_{z \in F(T)} \| x - z \|$.

***Theorem** 4.3* — *Let $(E, \| \cdot \|)$ be a Banach space with property $(U, \lambda, m+1+m)$, $\lambda \in I\!R$, $m \in I\!N$; $\phi \neq D \subseteq E$ be closed convex and bounded $T: D \to D$ is uniformly L-Lipschitzian for some $L > 0$ and asymptotically hemicontractive with sequence $\{k_n\}$ and also T satisfies Condition A. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ and sequence of real number satisfying the conditions (i), (ii) and (iii).*

Pick $x_1 \in D$ and define $x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n(y_n)$ and $y_n = (1 - \beta_n) x_n + T^n(x_n)$ for all $n \in I\!N$. If $\lim d(x_n, F(T))$ exists and $F(T)$ is closed, then the sequence $\{x_n\}$ converges strongly to an element of $F(T)$.

PROOF : By Condition A, we have for all $n \in I\!N$,

$$\| x_n - Tx_n \| \geq f(d(x_n, F(T))). \qquad \text{... (11)}$$

Since $\lim \| x_n - p \|$ exists for $p \in F(T)$, assign $\varepsilon > 0$. Then there exists an n_0 such that

$$\| x_n - p \| < \varepsilon/2 \text{ for all } n \geq n_0.$$

Hence, if n and m are both greater than n_0, for some $p \in F(T)$,

$$\| x_n - x_m \| \leq 2 \| x_{n_0} - p \|$$

This shows that $\{x_n\}$ is a Cauchy sequence in D and hence, there exists $z \in D$ with

$$\lim \| x_n - z \| = 0.$$

Now, from Theorem 4.1, we have

$$\lim \| x_n - Tx_n \| = 0. \qquad \text{... (12)}$$

Since $\lim d(x_n, F(T))$ exists, suppose that $\lim d(x_n, F(T)) = r > 0$. It follows from (11) that $\lim \| x_n - Tx_n \| \neq 0$, a contradiction so that $z \in F(T)$. Again, for any $p \in F(T)$

$$\| z - Tz \|^{m+1} \leq (\| z - p \| + \| Tp - Tz \|)^{m+1}$$

$$\leq 2^m (\| z - p \|^{m+1} + \| Tp - Tz \|^{m+1}). \qquad \text{... (13)}$$

Since T is uniformly L-Lipschitzian, then we have

$$\| Tp - Tz \| \leq L \| p - z \|.$$

Substitution of the above inequality into (13), gives :

$$\| z - Tz \|^{m+1} \leq 2^m (1 + L) \| z - p \|^{m+1}$$

implies

$$\| z - Tz \|^{m+1} \leq 2^m (1 + L)\, d(z, F(T)) = 0,$$

completing the proof.

***Theorem** 4.4 — Let $(E, \| \cdot \|)$ be a Banach space with property $(U, \lambda, m+1, m)$, $\lambda \in I\!R$, $m \in I\!N$; $\phi \neq D \subseteq X$ be closed convex and bounded; $T : D \to D$ is uniformly L-Lipschitzian for some $L > 0$ and asymptotically pseudo-contractive with sequence $\{k_n\}$ and also T is satisfies condition A. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the conditions (i), (ii) and (iii).*

Pick $x_1 \in D$ and define $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n(y_n)$ and $y_n = (1 - \beta_n) x_n + \beta_n T^n(x_n)$ for all $n \in I\!N$. If lim $d(x_n, F(T))$ exist and $F(T)$ is closed, then the sequence $\{x_n\}$ converges strongly to an element of $F(T)$.

PROOF : The result follows Theorem 4.3.

Remark : Theorem 4.4 improves theorem 2.3 of Schu[11].

REFERENCES

1. S. C. Bose, *Proc. Amer. math. Soc.* **68** (1978), 305-308.
2. S. C. Bose and S. K. Laskar, *J. math. phys. Sci.* **19** (1985), 503-509.
3. K. Deimling, *Manuscripta Math.* **13** (1974).
4. J. Diestel, *Geometry of Banach Spaces-Selected Topics*, L. Not *485. Springer-Verlag Berlin, Heidelberg, New York* 1975.
5. K. Goebel and W. A. Kirk, *Proc. Amer. math. Soc.* **35** (1972), 171-74.
6. J. P. Gossez and E. Lami Dozo, *Pacific J. Math.* **40** (1972), 565-73.
7. O. Hanner, *Arklv. Math.* **3** (19) (1956), 239-44.
8. Z. Opial, *Bull. Amer. math. Soc.* **73** (1967), 591-97.
9. B. E. Rhoades, *J. math. Anal. Appl.* **56** (1976), 741-50.
10. J. Schu, *Math. Nachr.* **153** (1991), 313-19.
11. J. Schu, *J. math. Anal. Appl.* **158** (1991), 407-13.
12. J. Schu, *Math. Machr.* **153** (1991), 313-19.
13. H. F. Senter and W. G. Dotson, Jr., *Proc. Amer. math. Soc.* **44** (1974), 375-80.
14. Yu. V.Trubmikov, *Sov. Math. (Izvestiy)* **31** (7) (1987), 74-83.