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ON THE GENERALIZED HANKEL TYPE INTEGRAL TRANSFORMATION OF GENERALIZED FUNCTIONS

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In this paper the generalized Hankel type integral transformation depending on three real parameters defined by

$F_1(y) = (F_{1,\mu,\alpha,\beta,\nu}f)(y) = \nu\beta y^{-1-2\alpha+2\nu} \int_0^\infty (xy)^\alpha J_\mu[\beta(xy)^\nu] f(x)dx \ (\mu \geq -1/2)$, where $J_\mu(x)$ is the Bessel function of the first kind of order μ, which reduces to almost all the Hankel, generalized Hankel and Hankel type integral transformations, is extended to certain spaces of generalized functions by kernel method in such a way that the theory of Koh and Zemanian in relation with Hankel transformation

$$F(y) = (h_\mu f)(y) = \int_0^\infty \sqrt{(xy)}\, J_\mu(xy) f(x)dx \ \ (\mu \geq -1/2)$$

appears then as a particular case for $\nu = 1, \beta = 1, \alpha = 1/2$ .... . An inversion theorem is established by interpreting convergence in the weak distributional sense. The theory thus developed is applied to solve certain initial value problems.

Key Words : Hankel Type Integral Transformation; Generalized Functions; Bessel Function; Testing Function Spaces

1. INTRODUCTION

Some generalizations of the classical Hankel transformation

$$F(y) = (h_\mu f)(y) = \int_0^\infty \sqrt{xy}\, J_\mu(xy) f(x)dx \ \ (\mu \geq -1/2) \qquad \text{... (1.1)}$$

were given by many authors from time to time. In previous paper [see Malgonde[6]] we introduced the generalized Hankel type transformation depending on three real parameters (α, β, ν) defined by

$$F_1(y) = (F_{1,\mu,\alpha,\beta,\nu}f)(y) = \nu\beta y^{-1-2\alpha+2\nu} \int_0^\infty (xy)^\alpha J_\mu[\beta(xy)^\nu] f(x)dx \quad (\mu \geq -2) \qquad \dots (1.2)$$

where α, β and ν are any arbitrary real italic numbers and $J_\mu(z)$ is the Bessel function of the first kind of order μ.

In view of the general nature of the kernel involved in F_1-transformation (1.2) on specializing the parameters we obtain the Hankel transformation [see Zemanian[17], Titchmarsh[14], Watson[15], Sneddon[13]], the Hankel-Schwartz transform [Schwartz[11]], the Hankel-Clifford transform [Mendez[8]], amongst others.

More recently a Hankel type integral transformation [see Linares and Mendez[4]] defined by

$$F(y) = (F_{\mu,\nu}f)(y) = y^{1+2\mu} \int_0^\infty (xy)^{-\mu} J_\mu(xy) f(x)dx, \qquad \dots (1.3)$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν $(\nu \geq -1/2)$ and μ is an arbitrary real parameter, is a particular case of (1.2) for $\beta = 1, \nu = 1, \alpha = -\mu, \mu = \nu$, has been extended to certain space of generalized functions by Malgonde[5].

As it is well known, there exist two ways to define integral transform of generalized functions, the adjoint and the kernel method. The adjoint method has been employed by Zemanian[17], Mendez[9], Lee[3], Schuitman[10], amongst others. The kernel method was used by Koh and Zemanian[2], Dube and Pandey[1], amongst others.

We shall use quite a few times the asymptotic expansions

$$J_\mu[\beta x^\nu] \cong \sqrt{\frac{2\nu}{\beta\pi}}\, x^{-\nu/2} \cos[\beta x^\nu - (\pi/2)(\mu + 1/2)] + O(\beta x^\nu)^{-3/2}, \text{ as } x \to \infty$$

and

$$J_\mu[\beta x^\nu] \cong [\Gamma(1+\nu)]^{-1} (\beta x^\nu/2)^\mu, \text{ as } x \to 0, \mu \geq -1/2 \qquad \dots (1.4)$$

and the following some of the important classical results [see Malgonde[6]].

Lemma 1.1 — (Inversion Formula) — If $f(x)$ is of bounded variation into a neighbourhood of the point $x = x_0 > 0$, $\mu \geq -1/2$ and the integral $\int_0^\infty |f(x)| x^{\alpha-\nu/2} dx$ exists, then

$$\lim_{R\to\infty} \nu\beta x_0^{-1-2\alpha+2\nu} \int_0^R (x_0 y)^\alpha J_\mu[\beta(x_0 y)^\nu] F_1(y)dydx = \frac{1}{2}[f(x_0+0) + f(x_0-0)]. \dots (1.5)$$

Lemma 1.2 — (Mixed Parseval's Equation) — If $f(x)x^{\alpha+\mu}$ and $F_2(y)\, y^{\mu\nu-\alpha-1+2\nu}$ are in $L_1(0,\infty)$, $F_1(y) = F_{1,\mu,\alpha,\beta,\nu}[f(x)](y)$

and

$$F_2(y) = F_{2,\mu,\alpha,\beta,\nu}[g(x)](y) = \nu\beta \int_0^\infty x^{-1-2\alpha+2\nu} (xy)^\alpha J_\mu[\beta(xy)^\nu] g(x)dx \qquad \dots (1.6)$$

then for $\mu \geq -1/2$

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_1(y)\, F_2(y)dy. \qquad \ldots (1.7)$$

According to Mendez[9] equality (1.7) is called the mixed Parseval's equation for the F_1-transformation or $F_{1,\mu,\alpha,\beta,v}$-transformation.

The F_1-transformation was recently extended by Malgonde and Bandewar[7] to certain generalized functions of slow growth through a generalization of mixed Parseval eq. (1.7).

In the present paper, we extend the F_1-transformation (1.2) to other spaces of generalized functions following a different procedure called the kernel method. Theorems on smoothness, boundedness, inversion and uniqueness, together with an operation-transform formula for a Bessel-type differential operator are presented.

The notation and terminology used here are those of Zemanian[17]. Throughout this work I denotes the open interval $(0, \infty)$, $D(I)$ denotes the space of smooth functions whose supports are compact subsets of I. We assign to $D(I)$ the topology that makes its dual $D'(I)$ the space of Schwartz's distributions on I [see Schwartz[12]]. $E(I)$ and $E'(I)$ are, respectively, the space of smooth functions on I and the space of distributions with compact supports on I. We use the following operators :

$$D = D_x = \frac{d}{dx},\ \Delta^k_{\alpha,\mu,v} = (x^{-\mu v+\alpha+1-2v} D_x x^{2\mu v+1} D_x x^{-\alpha-\mu v})^k \text{ and}$$

$$(\Delta^*_{\alpha,\mu,v})^k = (x^{-\alpha-\mu v} D x^{2\mu v+1} D x^{-\mu v+\alpha+1-2v})^k \text{ for } k = 0, 1, 2 \ldots .$$

Let α, μ and v be any arbitrary real numbers. $H_{1,\alpha,\mu,v}$ and $H_{2,\alpha,\mu,v}$ denote the linear spaces consisting of all complex-valued smooth functions $\phi(x)$ on I such that for every pair of non-negative integers (m, k), the numbers

$$\gamma^{1,\alpha,\mu,v}_{m,k}(\phi(x)) = \sup_{0<x<\infty} |x^m (x^{1-2v} D)^k x^{-\mu v+\alpha+1-2v}\ \phi(x)| \qquad \ldots (1.8)$$

and

$$\gamma^{1,\alpha,\mu,v}_{m,k}(\phi(x)) = \sup_{0<x<\infty} |x^m (x^{1-2v} D)^k x^{-\alpha-\mu v}\ \phi(x)| \qquad \ldots (1.9)$$

exist respectively. The set of seminorms $\left\{ \gamma^{i,\alpha,\mu,v}_{m,k} \right\}_{m,k=0}^{\infty}$ where $i = 1, 2$ generates the topology of $H_{1,\alpha,\mu,v}$ and $H_{2,\alpha,\mu,v}$ respectively. The duals of $H_{1,\alpha,\mu,v}$ and $H_{2,\alpha,\mu,v}$ are denoted by $H'_{1,\alpha,\mu,v}$ and $H'_{2,\alpha,\mu,v}$ respectively.

2. Testing Function Spaces $\mathbb{H}_{\alpha,\mu,v,a}$ and $\mathbb{H}_{\alpha,\mu,v}(\sigma)$ and Their Duals

Let a denote a positive real number and α, μ, v be any arbitrary real parameters. Then for each a, α, μ and v we define $\mathbb{H}_{\alpha,\mu,v,a}$ as the space of testing functions $\phi(x)$ defined on $0 < x < \infty$ and for which

$$\eta_k^{\alpha,\mu,\nu,a}(\phi) = \sup_{0<x<\infty} | e^{-ax} x^{-\alpha-\mu\nu} \Delta_{\alpha,\mu,\nu}^k \phi(x) | < \infty \quad \dots (2.1)$$

for $k = 0, 1, 2, \dots$.

We assign to $I\!H_{\alpha,\mu,\nu,a}$ the topology generated by the countable multinorm $\left\{ \eta_k^{\alpha,\mu,\nu,a} \right\}_{k=0}^{\infty}$. $I\!H_{\alpha,\mu,\nu,a}$ is Hausdorff space, since $\eta_0^{\alpha,\mu,\nu,a}$ is a norm on $I\!H_{\alpha,\mu,\nu,a}$. Moreover, $I\!H_{\alpha,\mu,\nu,a}$ is a locally convex linear space that it satisfies the first axiom of countability. The dual space $I\!H'_{\alpha,\mu,\nu,a}$ consists of all continuous linear functionals on $I\!H_{\alpha,\mu,\nu,a}$.

Following Koh and Zemanian[2], we now list some properties of these spaces.

(*i*) Let $\mu \geq -1/2, a > 0$ and α, ν be any arbitrary real parameters and $K_1(x, y) = \nu\beta y^{-1-2\alpha+2\nu} (xy)^{\alpha} J_{\mu} [\beta(xy)^{\nu}]$. For a fixed positive real italic number y,

$$\frac{\partial^m}{\partial y^m} [K_1 (x, y)] \in I\!H_{\alpha,\mu,\nu,a}, m = 0, 1, 2, \dots.$$

(*ii*) $I\!H_{\alpha,\mu,\nu,a}$ is sequentially complete and therefore a Fréchet space. Hence $I\!H'_{\alpha,\mu,\nu,a}$ is also sequentially complete.

(*iii*) If $a > b > 0$, then $I\!H_{\alpha,\mu,\nu,b} \subset I\!H_{\alpha,\mu,\nu,a}$, and the topology of $I\!H_{\alpha,\mu,\nu,b}$ is stronger than that induced on it by $I\!H_{\alpha,\mu,\nu,a}$.

(*iv*) $H_{2,\alpha,\mu,\nu}$ is a proper subset of $I\!H_{\alpha,\mu,\nu,a}$ for every choice of $a > 0$, and the topology of $H_{2,\alpha,\mu,\nu}$ is stronger than that induced on it by $I\!H_{\alpha,\mu,\nu,a}$.

(*v*) $D(I) \subset I\!H_{\alpha,\mu,\nu,a}$, and the topology of $D(I)$ is stronger than that induced on it by $I\!H_{\alpha,\mu,\nu,a}$.

(*vi*) For every choice of α, μ, ν and a, $I\!H_{\alpha,\mu,\nu,a} \subset E(I)$. Moreover, it is dense in $E(I)$ because $D(I) \subset I\!H_{\alpha,\mu,\nu,a}$ and $D(I)$ is dense in $E(I)$. The topology of $I\!H_{\alpha,\mu,\nu,a}$ is stronger than that induced on it by $E(I)$. Hence, $E'(I)$ can be identified with a subspace of $I\!H'_{\alpha,\mu,\nu,a}$.

(*vii*) The operation $\phi \rightarrow \Delta_{\alpha,\mu,\nu}\phi$ is a continuous linear mapping of $I\!H_{\alpha,\mu,\nu,a}$ into itself since

$$\eta_k^{\alpha,\mu,\nu,a} (\Delta_{\alpha,\mu,\nu}\phi) = \eta_{k+1}^{\alpha,\mu,\nu,a} (\phi) \text{ for } k = 0, 1, 2 \dots$$

(*viii*) Let f be an arbitrary element of $I\!H'_{\alpha,\mu,\nu,a}$. Then there exist bounded measurable functions $g_i(x)$ defined for $x > 0$ and $i = 0, 1, 2, \dots, r$, where r is some nonnegative integer depending upon f, such that for an arbitrary $\phi \in D(I)$ we have

$$\langle f, \phi \rangle = \langle \sum_{i=0}^{r} (\Delta^*_{\alpha,\mu,\nu})^i \{e^{-ax} x^{-\alpha-\mu\nu} (-D_x) g_i(x)\}, \phi(x) \rangle.$$

We turn now to the definition of a certain countable-union spaces $I\!H_{\alpha,\mu,\nu}(\sigma)$ that arise from the $I\!H_{\alpha,\mu,\nu,a}$spaces. Our subsequent discussion takes on a simpler form when the $I\!H_{\alpha,\mu,\nu}(\sigma)$ spaces are used in place of the $I\!H_{\alpha,\mu,\nu,a}$spaces.

Following Koh and Zemanian[2], $I\!H_{\alpha,\mu,\nu}(\sigma) = \bigcup_{p=1}^{\infty} I\!H_{\alpha,\mu,\nu,a_p}$ is the countable-union space where $\{a_p\}_{p=1}^{\infty}$ is a monotonic sequence of positive numbers such that $a_p \to \sigma\,(\sigma = +\infty$ is allowed). A generalized function f is $F'_{1,\mu,\alpha,\beta,\nu}$-transformable if $f \in I\!H'_{\alpha,\mu,\nu}(\sigma)$ for some $\sigma > 0$ where $I\!H'_{\alpha,\mu,\nu}(\sigma)$ is the dual of $I\!H_{\alpha,\mu,\nu}(\sigma)$.

In view of our definitions of $I\!H_{\alpha,\mu,\nu}(\sigma)$ and its dual, the following lemmas are immediate.

Lemma 2.1 — For any fixed $y > 0$, $\frac{\partial^m}{\partial y^m}[K_1(x,y)] \in I\!H_{\alpha,\mu,\nu}(\sigma)$, $m = 0, 1, 2, \ldots$ where $\sigma > 0$.

Lemma 2.2 — For every choise of $\sigma > 0, H_{2,\alpha,\mu,\nu} \subset I\!H_{\alpha,\mu,\nu}(\sigma)$, and convergence in $H_{2,\alpha,\mu,\nu}$ implies convergence in $I\!H_{\alpha,\mu,\nu}(\sigma)$. The restriction of $f \in I\!H'_{\alpha,\mu,\nu}(\sigma)$ to $H_{2,\alpha,\mu,\nu}$ is in $H'_{2,\alpha,\mu,\nu}$ and convergence in $I\!H'_{\alpha,\mu,\nu}(\sigma)$ implies convergence in $H'_{2,\alpha,\mu,\nu}$.

Lemma 2.3 — The operation $\phi \to \Delta_{\alpha,\mu,\nu}\phi$ is a continuous linear mapping of $I\!H_{\alpha,\mu,\nu}(\sigma)$ into itself. Hence the operation $f \to \Delta^*_{\alpha,\mu,\nu} f$ is a continuous linear mapping of $I\!H'_{\alpha,\mu,\nu}(\sigma)$ into itself [See Zemanian[17]].

As was indicated in vote (*vi*), $I\!H'_{\alpha,\mu,\nu}(\sigma)$ contains all distributions of compact support on $I = (0, \infty)$. Similarly, any conventional function f for some $a < \sigma$ is a member of $I\!H'_{\alpha,\mu,\nu}(\sigma)$, as is every generalized derivative $(\Delta^*_{\alpha,\mu,\nu})^k f, k = 1, 2, 3.$ according to Lemma 2.3. Moreover, we may say that the members of $I\!H'_{\alpha,\mu,\nu}(\sigma)$ are "generalized functions of exponential descent", since the italic multinorms $\left\{ \eta_k^{\alpha,\mu,\nu,a} \right\}$ show that the testing functions $\phi \in I\!H_{\alpha,\mu,\nu,a}$are at most of exponential growth.

3. Generalized Hankel Type Integral Transformation $F'_{1,\mu,\alpha,\beta,\gamma}$

Let $\mu \geq -1/2$ and α, ν be any arbitrary real numbers. In view of note (*iii*) Of § 2, to every $f \in I\!H'_{\alpha,\mu,\nu,a}$ there exists a unique real number σ_f (possibly, $\sigma_f = +\infty$) such that $f \in I\!H'_{\alpha,\mu,\nu,b}$ if $b < \sigma_f$ and $f \in I\!H'_{\alpha,\mu,\nu,b}$ if $b > \sigma_f$. Therefore, $f \in I\!H'_{\alpha,\mu,\nu}(\sigma_f)$. We define the μth order generalized Hankel type integral transform $F'_{1,\mu,\alpha,\beta,\nu}\ f$ of f as the application of f to the kernel $K_1(x,y)$; i.e.,

$$F_1(y) = (F'_{1,\mu,\alpha,\beta,\nu} f)(y) = \langle f(x), K_1(x,y) \rangle, \qquad \ldots (3.1)$$

where $0<y<\infty$ and $\sigma_f>0$. The right hand side of (3.1) is meaningful by Lemma 2.1 for each $y>0$ and $\sigma_f>0$.

Lemma 3.1 — Let a and σ_f be fixed real numbers such that $0<a<\sigma_f$.

For all fixed $y>0$, for $\mu\geq -1/2$ and for $0<x<\infty$

$$|e^{-ax}[\beta(xy)^{\nu}]^{-\mu}J_{\mu}[\beta(xy)^{\nu}]|<A_{\mu,\beta,\nu} \quad \ldots (3.2)$$

where $A_{\mu,\beta,\nu}$ is a constant with respect to x and y.

PROOF : The proof is simple and can be verified [see Koh and Zemanian[2]].

Theorem 3.1 — *(Analyticity of $F_1(y)$) : For $y>0$, let $F_1(y)$ be defined by (3.1). Then*

$$\frac{d}{dy}F_1(y)=\left\langle f(x),\frac{\partial}{\partial y}K_1(x,y)\right\rangle.$$

PROOF : The proof can be easily verified following Malgonde[5].

Theorem 3.2 *(Boundedness of $F_1(y)$) — Let $F_1(y)$ be defined by (3.1). Then $F_1(y)$ is bounded according to*

$$|F_1(y)|\leq\begin{cases} c\,y^{-1-\alpha+2\nu+\mu\nu} & \text{as } y\to 0+\\ c\,y^{2\nu r-1-\alpha+2\nu+\mu\nu} & \text{as } y\to\infty\end{cases} \quad \ldots (3.3)$$

where c is a positive constant and r is non-negative integer.

PROOF : Proof is very similar to that of Malgonde[5].

In view of note (*iv*) of § 2 and Lemma 2.2, if f is in $I\!H'_{\alpha,\mu,\nu}(\sigma_f)$ then f belongs to $H'_{2,\alpha,\mu,\nu}$ provided that $\mu\geq -1/2$. We now show that generalized Hankel type integral transform of $f\in I\!H'_{\alpha,\mu,\nu}(\sigma_f)$ given by (3.1) is equal (in the sense of equality in $H'_{2,\alpha,\mu,\nu}$) to the generalized Hankel type integral transform of f as defined [see Malgonde and Bandewar[7]]

by $$\langle F'_1 f,\phi\rangle=\langle f,F_2\phi\rangle \quad \ldots (3.4)$$

for every $f\in H'_{2,\alpha,\mu,\nu}$ and $\phi\in H_{2,\alpha,\mu,\nu}$.

Theorem 3.3 — *Let $f\in H'_{\alpha,\mu,\nu}(\sigma_f)$, $\phi\in I\!H_{2,\alpha,\mu,\nu}$ and $\mu\geq -1/2$. Then*

$$\langle\langle f(x),K_1(x,y)\rangle,\phi(y)\rangle=\langle f(x)\int_0^{\infty}K_1(x,y)\,\phi(y)\,dy\rangle. \quad \ldots (3.5)$$

PROOF : Proof follows on the similar lines as that of Malgonde[5].

Theorem 3.4 — *Let $F_1(y)=(F'_{1,\mu,\alpha,\beta,\nu}f)(y), f\in I\!H'_{\alpha,\mu,\nu}(\sigma_f)$ as in (3.1) where $y>0$. Let $\mu\geq -1/2$. Then, in the sense of convergence in $D'(I)$,*

$$f(x) = \lim_{R \to \infty} \int_0^R F_1(y)\, K_2(x, y)\, dy, \qquad \ldots (3.6)$$

where $\qquad K_2(x, y) = \nu\beta x^{-1-2\alpha+2\nu} (xy)^{\alpha} J_{\mu}[\beta(xy)^{\nu}].$

PROOF : Let $\phi(x) \in D(I)$. We wish to show that

$$\langle \int_0^R F_1(y)\, K_2(x, y) dy, \phi(x) \rangle \qquad \ldots (3.7)$$

tends to $\langle f(x), \phi(x) \rangle$ as $R \to \infty$. From the smoothness of $F_1(y)$ and the fact that support of $\phi(x)$ is a compact subset of I, we may write (3.7) as a repeated integral on (x, y) having a continuous integrand and a finite domain of integration. Hence we can change the order of integration and obtain

$$\int_0^{\infty} \phi(x) \int_0^R F_1(y)\, K_2(x, y)\, dy dx = \int_0^R \langle f(t), K_1(t, y) \rangle \int_0^{\infty} \phi(x)\, K_2\,(x, y)\, dx\, dy \qquad \ldots (3.8)$$

By an argument based on Riemann sums for the integral $\int_0^R .. dy$, the right side of (3.8) can be written as

$$\langle\ f(t), \int_0^R K_1(t, y) \int_0^{\infty} \phi(x)\, K_2(x, y)\, dx dy\ \rangle. \qquad \ldots (3.9)$$

The formula [see Malgonde[6]]

$$\nu\beta \int_0^R y^{-1+2\nu}\ J_{\mu}[\beta(ty)^{\nu}]\, J_{\mu}[\beta(xy)^{\nu}]\, dy$$

$$= \frac{R^{\nu}}{x^{2\nu} - t^{2\nu}} [x^{\nu} J_{\mu+1}[\beta(xR)^{\nu}]\, J_{\mu}[\beta(tR)^{\nu}] - t^{\nu} J_{\mu+1}[\beta(tR)^{\nu}]\, J_{\mu}[\beta(xR)^{\nu}]$$

and the asymptotic representations of the Bessel functions enable us to show that for any $a > 0$, the testing function in (3.9) converges in $I\!H_{\alpha, \mu, \nu, a}$ to $\phi(t)$ as $R \to \infty$. Since $f \in I\!H'_{\alpha, \mu, \nu, a}$ where $0 < a < \sigma_f$, it follows that (3.9) converges to $\langle f(t), \phi(t) \rangle$ as $R \to \infty$. This proves the theorem.

Theorem 3.5 (Uniqueness Theorem) — Let $F_1(y) = F'_{1, \mu, \alpha, \beta, \nu} f)\,(y)$ *for* $y > 0$ *and* $G_1(y) = (F'_{1, \mu, \alpha, \beta, \nu}\, g)\,(y)$ *for* $y > 0$*, f and g being in* $I\!H'_{\alpha, \mu, \nu}(\sigma)$*. If* $F_1(y) = G_1(y)$*, for every* $y > 0$*, then* $f = g$ *in the sense of equality in* $D'(I)$.

PROOF : By Theorem3.4, $f-g=\lim_{R\to\infty}\int_0^R [F_1(y)-G_1(y)]\,K_2(x,y)dy=0.$

4. AN OPERATION-TRANSFORM FORMULA

In this section, we shall apply the preceding theory in solving certain differential equations involving generalized functions. We define the operator

$\Delta^*_{\alpha,\mu,\nu} : I\!H'_{\alpha,\mu,\nu}(\sigma_f) \to I\!H'_{\alpha,\mu,\nu}(\sigma_f)$ by the relation

$$\langle \Delta^*_{\alpha,\mu,\nu} f(x), \phi(x) \rangle = \langle f(x), \Delta_{\alpha,\mu,\nu}\phi(x) \rangle \qquad \text{... (4.1)}$$

for all $f \in I\!H'_{\alpha,\mu,\nu}(\sigma_f)$ and $\phi \in I\!H_{\alpha,\mu,\nu}(\sigma_f)$, $\mu \geq -1/2$ and for α and ν arbitrary real numbers. It can be readily seen that

$$\langle (\Delta^*_{\alpha,\mu,\nu})^k f(x), \phi(x) \rangle = \langle f(x), \Delta^k_{\alpha,\mu,\nu}\phi(x) \rangle$$

for each k = 1, 2, 3, ... In case f is a regular distribution generated by an element of $D'(I)$, then

$$\Delta^*_{\alpha,\mu,\nu} = x^{-\alpha-\mu\nu} D x^{2\mu\nu+1} D x^{-\mu\nu+\alpha+1-2\nu}$$

$$= x^{2-2\nu}D^2-(4\nu-2\alpha-3)\,x^{1-2\nu}D-[\mu^2\nu^2-(\alpha+1-2\nu)^2]\,x^{-2\nu}.$$

The generalized Hankel type integral transformation is used in solving initial value problems. Indeed, we now establish a theorem that enables us to transform a differential equation of the form

$$P\,[\Delta^*_{\alpha,\mu,\nu}]\,u = g \qquad \text{... (4.2)}$$

where u and g possess $F'_{1,\mu,\alpha,\beta,\nu}$-transforms and P is any polynomial having no zeros on $-\infty < x \leq 0$ into an algebraic equation of the form

$$p\,[-\nu^2\beta^2 y^{2\nu}]\,U(y) = G(y)$$

where $U(y) = (F'_{1,\mu,\alpha,\beta,\nu}u(x))\,(y)$ and $G(y) = (F'_{1,\mu,\alpha,\beta,\nu}g(x))\,(y)$.

Theorem 4.1 — *For k = 0, 1, 2, ...*

$$F'_{1,\mu,\alpha,\beta,\nu}[\Delta^{*k}_{\alpha,\mu,\nu}f]\,(y) = (-\beta^2\nu^2)^k\,y^{2\nu k}\,F'_{1,\mu,\alpha,\beta,\nu}f \qquad \text{... (4.3)}$$

for every $f \in H'_{\alpha,\mu,\nu}(\sigma_f)$.

PROOF : From our definition of the operation $\Delta^*_{\alpha,\mu,\nu}$ defined in (4.1) and using the fact that $\Delta^k_{\alpha,\mu,\nu}[K_1(x,y)] = (-\nu^2\beta^2)^k\,y^{2\nu k}\,K_1(x,y)$, we have

$$F'_{1,\mu,\alpha,\beta,\nu}[\Delta^{*k}_{\alpha,\mu,\nu}f](y) = \langle \Delta^{*k}_{\alpha,\mu,\nu}f(x), K_1(x,y)\rangle = \langle f(x), \Delta^{k}_{\alpha,\mu,\nu}K_1(x,y)\rangle$$

$$= (-\beta^2\nu^2)^k y^{2\nu k}\langle f(x), K_1(x,y)\rangle$$

$$= (-\beta^2\nu^2)^k y^{2\nu k} F'_{1,\mu,\alpha,\beta,\nu}f.$$

Theorem 4.1 can be applied to the resolution of the generalized Cauchy's problem to determine the function $u(x, t)$ which satisfies

$$x^{2-2\nu}\frac{\partial^2 u}{\partial x^2} - (4\nu - 2\alpha - 3)x^{1-2\nu}\frac{\partial u}{\partial x} - [\mu^2\nu^2 - (\alpha + 1 - 2\nu)^2]\,x^{-2\nu}u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \dots (4.4)$$

for $\mu \geq -1/2$, α, ν are real numbers and $c > 0$ in Malgonde and Bandewar[7] where the initial condition is now allowed to be a generalized function in $IH'_{\alpha,\mu,\nu}(\sigma)$ for some $\sigma > 0$. The method of solution parallels that of Malgonde[5] and Malgonde and Badewar[7].

Remark 1 : The transformation $F'_{1,\mu,\alpha,\beta,\nu}$ - can be also applied to find the solution $u(x,t) \in IH'_{\alpha,\mu,\nu}(\sigma)$ for some $\sigma > 0$ of the more general problem

$$\frac{\partial u}{\partial t} = P(\Delta^{*}_{\alpha,\mu,\nu})u,\; u(x,0) = f(x)$$

(A problem posed by Lee[3] when $\alpha = \mu + 1$, $\nu = \beta = 1$), where P denotes a polynomial with constant coefficients and f is a known member of $IH'_{\alpha,\mu,\nu}(\sigma)$.

Remark 2 : When $\mu = 0$, $\alpha = 1$, $\nu = \beta = 1$, (4.4) is a Cauchy problem for cylindrical waves, whose solution can be directly found, making unnecessary the previous change of variables done in Zemanian[17].

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