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A NOTE ON THE TORSIONAL BODY FORCES IN A VISCOELASTIC HALF-SPACE

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A mathematical treatment is presented for torsional disturbances in viscoelastic medium with a body force. Using the Hankel and Fourier cosine transforms, the disturbance is evaluated for a particular type of body force located over a plane at a constant depth from the surface. Numerical results are obtained for two special cases and are exhibited graphically.

Key Words : Torsional Oscillations; Visco-elastic Medium; Body Force; Hankel Transform

1. INTRODUCTION

The problem pertaining to the determination of stresses and displacements in an isotropic, homogeneous elastic medium due to a torsional oscillation has been a subject of considerable interest in solid mechanics and applied mathematics. Particularly torsional vibration of some anisotropic/viscoelastic material is very much used in measuring shear constant, shear velocities and shear elasticity of liquids.

As a fundamental example of mixed boundary value problems in elasticity, it has first attracted the attention of Reissner and Sagoci[1] and Sagoci[2] which involve the investigation of torsional oscillations, in an elastic half space under the action of periodic shear stress applied to a circular portion of the surface of the half-space, Since then, important contributions have also been made as in Collins[3], Eason[4] and Sneddon[5] to the stress analysis of this kind involving a semi-infinite medium as well as an elastic stratum. Erguven[6] has considered the dynamical Reissner-Sagocci problem for a radially non-homogeneous material, Pal and Kumar[7] have shown the effect of inhomogeneity on torsional impulsive motion over a circular region in a transversely isotropic elastic half-space. In another paper, Pal and Kumar[8] have considered the generation and propagation of SH-waves due to stress discontinuity in a linear viscoelastic layered medium. Recently existence and propagation of torsional surface waves in viscoelastic medium have been discussed by Dey *et al*[9].

In this paper, the problem of torsional disturbance due to a decaying body forces within a viscoelastic medium of Voigt type is being considered. Using the Hankel and Fourier cosine transforms, the solution is obtained for two particular type of body forces located over a plane at a constant depth from the plane face. Numerical results are obtained for Voigt solid and compared with elastic solid. Variations of disturbance are exhibit graphically for different radial distances and a particular value of depth from the surface.

2. Formulation of the Problem

Let (r, θ, z) be cylindrical polar co-ordinates. The origin is taken to be any point of the boundary of the half-space and z-axis is taken in the vertically downward directions. The material of the medium is taken to be homogeneous, isotropic and visco-elastic of Voigt type. Medium contains a body force which is torsional in nature and as a function r, z and t. For torsional disturbances the stress-strain relations as

$$\tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = \tau_{rz} = 0$$

$$\tau_{r\theta} = 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) e_{r\theta} = \left(\mu + \mu' \frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right),$$

and
$$\left(\mu + \mu' \frac{\partial}{\partial t}\right) \tau_{\theta z} = \left(\mu + \mu' \frac{\partial}{\partial t}\right) \frac{\partial v}{\partial r},$$

μ = Lames constant and μ' = viscoelastic parameter,

$\tau_{r\theta}$, $\tau_{\theta z}$ are shear stresses in the solid. Here the motion is symmetrical about z-axis so

$u_r = u_z = 0$ and $u_\theta(r, z, t)$ (say), and is independent of θ and is circumferential component of displacement.

The only non-zero equation of motion in terms of stresses is

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} + \frac{\partial \tau_{\theta z}}{\partial z} + F_\theta = \rho \frac{\partial^2 v}{\partial t^2} \quad \text{... (1)}$$

F_θ is the body force.

We take body force F_θ in the form

$$F_\theta = F(r, z) e^{-\omega t}, \ \omega > 0 \quad \text{... (2)}$$

The equation of motion (1) in terms of displacement component v may be written as

$$\left(\mu + \mu' \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right) + F(r, z)\, e^{-\omega t}\, \rho \frac{\partial^2 v}{\partial t^2}$$

Let us take $v(r, z, t) = f(r, z)\, e^{-\omega t}$, $\omega > 0$ and so final equation may be written as

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 f}{\partial z^2} - \left\{\frac{1}{r^2} + \frac{\rho \omega^2}{(\mu - \mu' \omega)}\right\} f + \frac{F}{(\mu - \mu' \omega)} = 0 \quad \text{... (3)}$$

3. Method of Solution

Eq. (3) will be solved using Hankel transform and Fourier cosine transform. The pasir of transformation is defined as

$$(f_i, F_i) = \int_{r=0}^{\alpha} (f, F) r J_1(\zeta r) dr$$

$$(f_2, F_2) = \sqrt{\frac{2}{\pi}} \int_{z=0}^{\alpha} (f_1, F_1) \cos(\zeta z) dz,$$

where $J_1(\zeta, r)$ is Bessel function of first kind and order one.

Now applying Hankel transform to eq. (3), we have

$$\frac{\partial^2 f}{\partial z^2} - \left[\zeta^2 + \frac{\rho\omega^2}{(\mu - \mu'\omega)} \right] f_1 + \frac{F_1}{(\mu - \mu'\omega)} = 0. \qquad \text{... (5)}$$

The initial and boundary conditions of the problem require that :

$$v(r, z, t) = \frac{\partial v}{\partial t}(r, z, t) = 0 \text{ at } t = 0 \qquad \text{... (6)}$$

and the plane boundary is stress free so

$\frac{\partial F_1}{\partial z} = 0$ on $z = 0$. We may assume that $\frac{\partial f_1}{\partial z} \to 0 \; z \to \alpha$.

Now applying fourier cosine transform to (5), we have

$$f_2(\xi, \zeta) = \frac{F_2}{(\mu - \mu'\omega)\left[\xi^2 + \zeta^2 \frac{\rho\omega^2}{(\mu - \mu'\omega)} \right]}$$

Hence, by inversion theorem on Hankel and Fourier transform, we have finally $v(r, z, t)$ is given by $v(r, z, t) = e^{-\omega t} f(r, z)$

$$= \frac{e^{-\omega t}}{(\mu - \mu'\omega)} \sqrt{\frac{2}{\pi}} \int_{\xi=0}^{\alpha} \int_{\zeta=0}^{\alpha} \frac{\xi F_2(\xi, \zeta) J_1(\xi, r) \cos(\xi z)\, d\xi\, d\zeta}{(\mu - \mu'\omega)\left[\xi^2 + \zeta^2 + \frac{\rho\omega^2}{(\mu - \mu'\omega)} \right]} \qquad \text{... (7)}$$

4. Decaying Torsional Body Force

To determine F_2 in (7), we assume that the body force acts inside the half space at a depth $z = h$ (> 0).

The body force is defined as

$$F_\theta = f(r, z) e^{-\omega t} = p(r)\, \delta(z - h) e^{-\omega t}. \qquad \text{... (8)}$$

Hence, $$f_1(\xi, z) = \int_{r=0}^{\alpha} p(r)\, \delta(z-h)\, J_1(\xi r) dr$$

and $$F_2(\xi, \zeta) = \sqrt{\frac{2}{\pi}} \int_{\xi=0}^{\alpha} p_1(\xi)\, \delta(z-h) \cos(\xi z) dz$$

$$= \sqrt{\frac{2}{\pi}}\, p_1(\xi) \cos(\zeta h),$$

where $$p_1(\xi) = \int_{r=0}^{\alpha} p_1(r)\, J_1(\xi r)\, dr$$

Hence finally,

$$v(r, z, t) = \frac{1}{\pi} \frac{e^{-\omega t}}{(\mu - \mu' \omega)} \int_{\xi=0}^{\alpha} = 0 \quad P_1(\xi) \left[\frac{\pi}{2n} e^{-n(z+h)} + \frac{\pi}{2n} e^{-n(z-h)} \right] \xi J_1(\xi r)\, d\xi. \qquad \ldots (9)$$

where

$$n = \sqrt{\xi^2 + \eta^2} \text{ and}$$

$$\eta + \sqrt{\frac{p\omega^2}{(\mu - \mu' \omega)}}$$

5. Special Cases

1. Let us assume that $p(r)$ the radial dependence of applied force be defined as

$$p(r) = \begin{cases} Q/r, & r \le a \\ 0 & r > a \end{cases}$$

Q and a are constant than $p_1(\xi) = \int_0^a Q\, J_1(\xi r) dr$

$$= \frac{Q}{\xi}(1 - J_0(\xi a)) \qquad \ldots (10)$$

Hence, from (9), the displacement $v(r, z, t)$ is given as

$$v(r, z, t) = \frac{Qe^{-\omega t}}{2(\mu - \mu' \omega)} \int_{\xi=0}^{\alpha} \frac{e^{-n(z+h)} + e^{-n(z-h)}}{n} J_1(\xi r)\, (1 - j_0(\xi a))\, d\xi \qquad \ldots (11)$$

2. Let $$p(r) = \begin{cases} Qr/(a^2 - 2)^{1/2}, & r < a \\ 0 \quad , & r > a \end{cases},$$

Q again a constant. This shows that the body force is in the form of angular displacement over a circular area of radius a and zero elsewhere. According to Sneddon[10]

$$p_1(\zeta) = Q \int_0^a \frac{r^2 J_1(\xi r)\, dr}{(a^2 - r^2)^{1/2}}$$

$$= Qa\left(\frac{a\pi}{2\xi}\right)^{1/2} J_1(a\xi),$$

In this case $$v(r, z, t) = \frac{Qe^{\omega t}}{2(\mu - \mu'\omega)} \int_{r=0}^{\alpha} \frac{e^{-n(z+h)} + e^{-n(z-h)}}{n} \frac{1}{\sqrt{\xi}} J_1(\xi r)\, j_1(\xi a)\, d\xi \quad \ldots (12)$$

6. Numerical Results and Discussions

For Voigt solid, we assume the viscoelastic parameter

$\eta = \dfrac{\rho\omega^2}{\mu - \mu'\omega}$ and $a = 1$ hence the integral for torsional disturbances are given by

$\dfrac{v(\mu - \mu'\omega)e^{\omega t}}{Q} = I$ (say) (on surface $z = 0$)

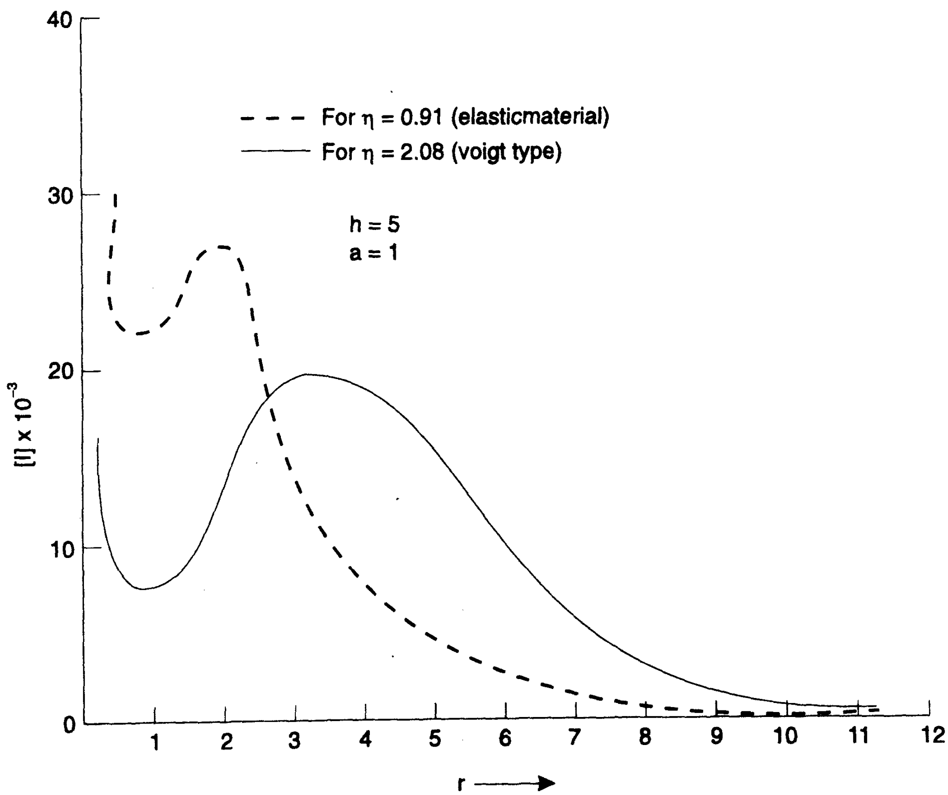

Fig 1. Variation of torsional disturbance with radial distance (special case 1)

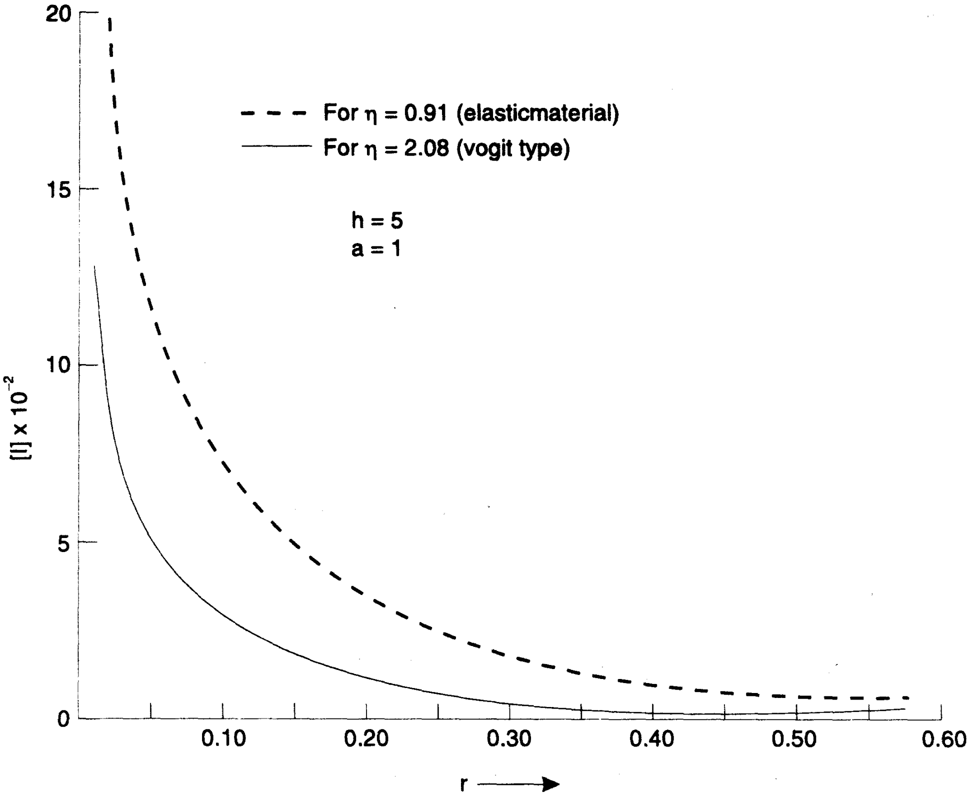

FIG 2. Variation of torsional disturbance with radial distance (special case 2)

where

$$[I] = \int_0^{\alpha} \frac{\cosh \sqrt{(\xi^2 + \eta^2)}\, h}{\sqrt{(\xi^2 + \eta^2)}} J_1(\xi r)\,(1 - J_0(\xi))\, d\xi \quad \ldots (13)$$

for special case (1) and

$$[I] = \int_0^{\alpha} \frac{2 \cosh \sqrt{(\xi^2 + \eta^2)\, h}}{\eta \sqrt{\xi}} J_1(\xi r)\, J_1(\xi))\, d\xi \quad \ldots (14)$$

for special case (2).

The values of I for different radial distance are calculated from eqs. (13) and (14) for a particular value of $h = 5$ (depth). The values of η for elastic material and for Viscoelastic material are taken from Bland[11]. The integrals (13) and (14) are evaluated using the known result of Watson[12]. The numerical results are depicted in Figs. 1 and 2, $\eta = 0.91$ for isotropic elastic material and $\eta = 2.08$ for viscoelastic material. From Fig. 1 it is clear that for $r > 1$, the torsional disturbance is infinite at the initial instant, then decreases and again increases giving rise to an oscillation and finally decreases steadily to zero. The trend remains the same for both values of η. From Fig. 2, it is inferred that for $r < 1$, the torsional disturbance decreases steadily from infinity at the initial instant towards zero. The nature remains the smae for both values of η. From Fig. 1, it is also clear the oscillation occur at different values of r. It is also obvious that the torsional disturbances produced by different kinds of body forces in a viscoelastic (Voight type) material are quite different in character.

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