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INTERACTING RADIATION FIELDS*

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The interaction of a Petrov type N gravitational field with a null electromagnetic field is considered and a metric describing such situation is obtained. The common propagation vector of these interacting pure radiation fields is geodetic, shear-free, twist-free and non-expanding and it is seen that along this propagation vector such pure radiation fields do not admit Ricci collineation and motion but admit Maxwell collineation. The Lanczos potential for these fields has also been calculated.

Key Words : Radiation Fields; Interaction of Fields; Petrov-type *N* Gravitational Field; Ricci Collineation; Motion; Maxwell Collineation; Lanczos Potential

1. INTRODUCTION

Since it was initiated some thirty years back, Petrov type N solutions of Einstein vacuum field equations are among the most interesting, but rather difficult and little explored of all empty space matrics[1-2]. From the physical point of view they represent space-time filled up entirely with gravitational radiation, while mathematically they form a class of solutions of Einstein equations which should be possible to be determined explicitly. The behaviour of gravitational radiation from a bounded source is an important physical problem. Even reasonably far from the source, however, twisting type N solutions of the vacuum field equations are required for an exact description of that radiation. Such solutions would provide small Laboratories in which to understand better the complete nature of singularities of type N solutions, and could also be used to check numerical solutions that include gravitational radiation.

While investigating the symmetries of the space-times, it has been shown by Collinson[3] that the only curvature collineation (*cc*) admitted by a vacuum space-time not of Petrov type N are conformal motion, but that type N vacuum spaces do admit *cc* which are not conformal motion. Collinson and French[4] have shown that the conformal motion admitted by type N space-time with twist-free geodesic ray is not homothetic. The only type N fields that admit conformal motion are *pp* waves[5] and the relevance of the homothetic vectors to the study of type N solutions has been very well described by McIntosh[6, 7]. Type N vacuum spaces that admit an expanding and/or twisting congruence and a homothetic motion are investigated by Halford[8], and recently, the expanding type N vacuum space-times admitting one homothetic and one Killing vector fields have been discussed by Plebanski and Przanoski[9]. Tariq and Tupper[10], McIntosh and Halford[11] and Hall[12], among many others, have considered type N gravitational fields and null electromagnetic fields. Most recently

*Dedicated to Professor S. Izhar Husain (author's research guide) on his 4th death anniversary on May 20, 1998.

Finsley III *et al.*[13] have found the Lie symmetries for twisting solutions of type N and further generalize this work to the case when the field equations have a particular non-vacuum source, created by a pressureless, null fluid or radiation field. All of them have discussed the symmetries of the space-time but they to do not find the solution of the coupled Einstein-Maxwell fields.

In this paper, we consider the free gravitational field to be the transverse gravitational wave zone which can be identified as Petrov type N fields[14] and we focus our attention on the interaction of pure electromagnetic radiation field and the pure gravitational field. For the sake of brevity we call these interacting radiation fields an PR fields. The Newman-Penrose formalism[15] is used to obtain a metric describing the behaviour of the PR fields. It is assumed that the reader is familiar with NP formalism and thus the equations taken directly from reference[15] shall be prefixed by NP. The vector l^{μ} of the null tetrad $\{l^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu}\}$ is taken as the common propagation vector of the two fields. By a propagation vector, we mean a repeated principal null vector.

We shall first give some basic remarks that are necessary for later study. Section II contains all NP equations, the commutator relations, the Binaitalic chi identities and the Maxwell equations for PR fields and their simplifications, while the space-time solutions for PR fields and its properties are discussed in the last section.

A vacuum metric contains a geodesic ray if and only if there is a principal null direction of the curvature tensor such that

$$l_{[\mu} R_{\alpha]\beta_{\gamma}[\delta}l_{\nu]} l^{\beta} l^{\nu} = 0, \qquad \text{... (1.1)}$$

i.e., tangent to a congruence of null geodesics

$$l_{\mu;\nu} l^{\nu} = 0, \; l^{\mu} l_{\mu} = 0. \qquad \text{... (1.2)}$$

Assume that the geodesics rays are hyper-surface orthogonal i.e., $l_{\mu} = u_{,\mu}$. Introduce the coordinates $u = x^1, r = x^2$ and x^3, x^4, where r is an affine parameter* $r_{,\mu} l^{\mu} = 1$) along the null geodesic and x^3, x^4 label the geodesic on each surface u = constant. The metric has the form

$$g^{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & g^{22} & g^{23} & g^{24} \\ 0 & g^{23} & g^{33} & g^{34} \\ 0 & g^{24} & g^{34} & g^{44} \end{bmatrix} \qquad \text{... (1.3)}$$

The null vectors $l^{\mu}, n^{\mu}, m^{\mu}$ have the form

$$l^{\mu} = \delta^{\mu}_{2}, \; n^{\mu} = \delta^{\mu}_{1} + U\,\delta^{\mu}_{2} + X^{i}\,\delta^{\mu}_{1}, \; m^{\mu} = \omega\delta^{\mu}_{2} + \xi^{i}\,\delta^{\mu}_{i}. \qquad \text{... (1.4)}$$

The metric components are

$$g^{22} = 2(U - \omega\bar{\omega}), \; g^{2i} = X^{i} - (\xi^{i}\bar{\omega} + \xi^{i}\omega), \; g^{ij} = -(\xi^{i}\bar{\xi}^{j} + \bar{\xi}^{i}\xi^{j}), \; i, j = 3, 4, \ldots \qquad \text{... (1.5)}$$

and the intrinsic derivative operators are

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* An affine parameter is a parameter along the geodesic such that equation of the geodesic takes the standard form.

$$D=\frac{\partial}{\partial r},\ \Delta=U\frac{\partial}{\partial r}+\frac{\partial}{\partial u}+X^i\frac{\partial}{\partial x^i},\ \delta=\omega\frac{\partial}{\partial r}+\xi^i\frac{\partial}{\partial x^i} \quad \text{... (1.6)}$$

The form of $g^{\mu\nu}$ is invariant under the following coordinate transformation:

$$r'=r+R(1,3,4),\ u'=u,\ \ x^{i'}=x^i. \quad \text{... (1.7 a)}$$

(shifts the origin)

$$\gamma'=r/\dot{\gamma},\ u'=\gamma(u),\ x^{x}=x^i. \quad \text{... (1.7 b)}$$

(relables hypersurface)

$$r'=r,\ u'=u,\ x^{i\prime}=x^i\,(1,3,4). \quad \text{... (1.7 c)}$$

(relables hypersurface)

The orthogonality of the tetrad vectors does not get affected by the transformation:

$$l^{\mu'}=l^{\mu},\ n^{\mu'}=n^{\mu}+\bar{P}\,m^{\mu}+\bar{P}m^{\mu}+P\,\bar{P}\,l^{\mu},\ m^{\mu'}=m^{\mu}+Pl^{\mu}, \quad \text{... (1.8 a)}$$

and $$l^{\mu\prime}=l^{\mu},\ n^{\mu\prime}=n^{\mu},\ m^{\mu\prime}=m^{\mu}\,e^{iQ} \quad \text{... (1.8 b)}$$

where P is a complex scalar and Q is a real independent of r. We shall also use the following coordinate freedom :

$$r'=r/\dot{r},\ u'=\gamma(u),\ \zeta'=x^{3'}+ix^{4'}=f(\zeta,u), \quad \text{... (1.9)}$$

where $\zeta=x^3+ix^4$.

The Weyl scalar characterizing a pure (null) gravitational field with propagation vector l^{μ} is[15]

$$\psi_4=\psi\neq 0,\ \psi_i=0,\ i=0,1,2,3. \quad \text{... (1.10)}$$

The Maxwell scalar characterizing a pure (null) electromagnetic field with propagation l^{μ} is[16]

$$\phi_2=\phi\neq 0,\ \phi_i=0,\ i=0,1. \quad \text{... (1.11)}$$

Since we are working with *PR* fields, therefore, from (1.10) and the Goldberg-Sachs theorem, the propagation vector l^{μ} is geodesic and shear-free; while from (1.11) we conclude that l^{μ} is scaled. In terms of the spin-coefficients this amount to the vanishing of k, σ and ε, i.e.,

$$k=\sigma=\varepsilon=0. \quad \text{... (1.12)}$$

II. Equations Describing the *PR* Fields and Their Simplifications

Under (1.10)-(1.12), the field equations (*NP* 4.2), the commutator relations (*NP* 4.4), the Bianchi identities (*NP* 4.5), the Maxwell equations (*NP A1*) and the coupled Bianchi identities[17] have the form

$$D\rho = \rho^2, \quad \text{... (2.1 } a)$$

$$D\tau = (\tau + \bar{\pi})\rho, \quad \text{... (2.1 } b)$$

$$D\alpha = \rho(\alpha + \pi), \quad \text{... (2.1 } c)$$

$$D\beta = \bar{\rho}\beta, \quad \text{... (2.1 } d)$$

$$D\gamma = (\tau + \bar{\pi})\,\alpha + (\bar{\tau} + \pi)\,\beta + \tau\pi, \quad \text{... (2.1 } e)$$

$$D\lambda - \bar{\delta}\pi = \rho\lambda + \pi^2 + (\alpha - \bar{\beta})\,\pi, \quad \text{... (2.1 } f)$$

$$D\mu - \delta\pi = \bar{\rho}\mu + \pi\bar{\pi} - \pi(\bar{\alpha} - \beta), \quad \text{... (2.1 } g)$$

$$D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\,\pi, \quad \text{... (2.1 } h)$$

$$\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu})\lambda - 3(\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\,\nu - \psi, \quad \text{... (2.1 } i)$$

$$\delta\rho = \rho(\bar{\alpha} + \beta) + (\rho - \bar{\rho})\tau, \quad \text{... (2.1 } j)$$

$$\delta\alpha - \bar{\delta}\beta = \rho\mu + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}), \quad \text{... (2.1 } k)$$

$$\delta\lambda - \bar{\delta}\mu = (\mu - \bar{\mu})\pi + \mu\,(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - (\rho - \bar{\rho})\nu, \quad \text{... (2.1 } l)$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + (\gamma + \bar{\gamma})\,\mu - \bar{\nu}\pi + (\tau - 3\beta - \bar{\alpha})\,\nu + \phi\bar{\phi}, \quad \text{... (2.1 } m)$$

$$\delta\gamma - \Delta\beta = (\tau - \bar{\alpha} - \beta)\gamma + \mu\tau - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda}, \quad \text{... (2.1 } n)$$

$$\delta\tau = \bar{\lambda}\rho + (\tau + \beta - \bar{\alpha})\,\tau, \quad \text{... (2.1 } o)$$

$$\Delta\rho - \bar{\delta}\tau\,t - \rho\bar{\mu} = (\bar{\beta} - \alpha - \bar{\tau}) + (\gamma + \bar{\gamma})\,\rho, \quad \text{... (2.1 } p)$$

$$\Delta\alpha - \bar{\delta}\gamma = \rho\nu - (\tau + \beta)\,\lambda - (\bar{\gamma} - \mu)\,\alpha + (\bar{\beta} - \bar{\tau})\,\gamma, \quad \text{... (2.1 } q)$$

$$(\Delta D - D\Delta)\eta = [(\gamma + \bar{\gamma})\,D - (\tau + \bar{\pi})\,\bar{\delta} - (\bar{\tau} + \pi)\,\delta]\,\eta, \quad \text{... (2.2 } a)$$

$$(\delta D - D\delta)\eta = [(\overline{\alpha} + \beta - \pi)\, D - \overline{\rho}\, \delta]\, \eta, \qquad \dots (2.2\ b)$$

$$(\delta \Delta - \Delta \delta)\, \eta = [-\, \overline{\nu} D + (\tau - \overline{\alpha} - \beta)\, \Delta + \lambda \overline{\delta} + (\mu - \gamma + \overline{\gamma})\, \delta]\, \eta, \qquad \dots (2.2\ c)$$

$$(\overline{\delta}\, \delta - \delta\, \overline{\delta})\eta = [(\overline{\mu} - \mu)\, D - (\overline{\alpha} - \beta)\, \overline{\delta} - (\overline{\beta} - \alpha)\, \delta]\, \eta, \qquad \dots (2.2\ d)$$

$$D\psi = \rho\psi, \qquad \dots (2.3\ a)$$

$$\delta\psi = -\, 4\beta\psi. \qquad \dots (2.3\ b)$$

$$D\phi = \rho\phi, \qquad \dots (2.4\ a)$$

$$\delta\phi = (\tau - 2\beta)\, \phi. \qquad \dots (2.4\ b)$$

$$D\phi\overline{\phi} = -\, \rho\phi\overline{\phi} \qquad \dots (2.5\ a)$$

and $$\delta(\phi\overline{\phi}) - \delta\psi = (-\, \tau + 4\beta)\, \psi - (2\, \overline{\beta} + 2\alpha)\, \phi\overline{\phi}. \qquad \dots (2.5\ b)$$

When the intrinsic derivatives (1.6) act on (2.2) we obtain the following relations between the tetrad components and the spin-coefficients [replacing η by u, r, x^3, x^4 on each eq. (2.2)] :

$$DU = -\, (\gamma + \overline{\gamma}) + (\tau + \overline{\pi})\, \overline{\omega} + (\overline{\tau} + \pi)\, \omega, \qquad \dots (2.6\ a)$$

$$DX^i = (\tau + \overline{\pi})\, \overline{\xi}^i + (\overline{\tau} + \pi)\, \xi^i, \qquad \dots (2.6\ b)$$

$$D\omega = -\, (\overline{\alpha} + \beta - \overline{\pi}) + \overline{\rho}\omega, \qquad \dots (2.6\ c)$$

$$D\xi^i = \overline{\rho}\xi^i, \qquad \dots (2.6\ d)$$

$$\tau = \overline{\alpha} + \beta \text{ [taking } \eta = u \text{ in (2.2 } c)], \qquad \dots (2.6\ e)$$

$$\delta U' - \Delta\omega = \overline{\nu} + (\tau - \overline{\alpha} - \beta)\, U + \overline{\lambda}\, \overline{\omega} + (\mu - \gamma + \overline{\gamma})\, \omega, \qquad \dots (2.6\ f)$$

$$\delta X^i - \Delta\xi^i = \lambda\overline{\xi}^i + (\overline{\tau} - \overline{\alpha} - \beta)\, X^i + (\mu - \gamma + \overline{\gamma})\, \xi^i, \qquad \dots (2.6\ g)$$

$$\rho = \overline{\rho}, \text{ [taking } \eta = u \text{ in (2.2 } d)], \qquad \dots (2.6\ h)$$

$$\overline{\delta}\omega - \delta\overline{\omega} = (\overline{\mu} - \mu) - (\overline{\alpha} - \beta)\, \overline{\omega} - (\overline{\beta} - \alpha)\, \omega \qquad \dots (2.6\ i)$$

and $$\overline{\delta}\xi^i - \delta\overline{\xi}^i = -\, (\overline{\alpha} - \beta)\, \overline{\xi}^i - (\overline{\beta} - \alpha)\xi^i. \qquad \dots (2.6\ j)$$

In our coordinate system, from (2.6 e), $\tau^0 = \overline{\alpha}^0 + \beta^0$, τ^0 may be set equal to zero by (1.8) and hence

$$\alpha = -\bar{\beta},\ \tau = 0. \qquad \text{... (2.7)}$$

Using (2.6 *h*) and (2.7), eqs. (2.1 *b*), (2.1 *j*) and (2.1 *o*) give rise to

$$\pi = 0, \qquad \text{... (2.8 } a)$$

$$\delta\rho = 0 \Rightarrow \omega \frac{\partial \rho}{\partial r} = \omega D\rho = 0 \Rightarrow \omega = 0 \qquad \text{... (2.8 } b)$$

and $\lambda = 0.$ ... (2.8 *c*)

From eqs. (2.6 *h*), (2.7) and (2.8), the radial (equations depending on *r*) and non-radial equations, after simplification, are as follows :

Radial Equations

$$D\rho = \rho^2, \qquad \text{... (2.9 } a)$$

$$D\alpha = \rho\alpha, \qquad \text{... (2.9 } b)$$

$$D\beta = \rho\beta, \qquad \text{... (2.9 } c)$$

$$D\gamma = 0, \qquad \text{... (2.9 } d)$$

$$D\mu = \rho\mu, \qquad \text{... (2.9 } e)$$

$$D\nu = 0, \qquad \text{... (2.9 } f)$$

$$D\psi = \rho\psi, \qquad \text{... (2.9 } g)$$

$$D\phi = \rho\phi, \qquad \text{... (2.9 } h)$$

$$DU = -(\gamma + \bar{\gamma}), \qquad \text{... (2.9 } i)$$

$$DX^i = 0 \qquad \text{... (2.9 } j)$$

and $D\xi^i = \rho\xi^i.$ ... (2.9 *k*)

Non-Radial Equations

$$\bar{\delta}\nu = -2\alpha\nu + \psi, \qquad \text{... (2.10 } a)$$

$$\delta\rho = 0, \qquad \text{... (2.10 } b)$$

$$\delta\alpha + \overline{\delta\alpha} = \rho\mu + 4\alpha\bar{\alpha}, \qquad \text{... (2.10 } c)$$

$$\bar{\delta}\mu = 0, \qquad \text{... (2.10 } d)$$

$$\delta\nu - \Delta\mu = \mu^2 + (\gamma + \bar{\gamma})\,\mu + 2\,\bar{\alpha}\nu + \phi\bar{\phi}, \qquad \text{... (2.10 } e)$$

$$\delta\gamma + \Delta\bar{\alpha} = \bar{\alpha}\,(\gamma - \bar{\gamma} - \mu), \qquad \text{... (2.10 } f)$$

$$\Delta\rho = -\,\rho\bar{\mu} + (\gamma + \bar{\gamma})\,\rho, \qquad \text{.. (2.10 } g)$$

$$\Delta\alpha - \bar{\delta}\gamma = \rho\nu + (\bar{\gamma} - \gamma - \mu)\alpha, \qquad \text{... (2.10 } h)$$

$$\delta\psi = 4\bar{\alpha}\psi, \qquad \text{... (2.10 } i)$$

$$\delta\phi = 2\bar{\alpha}\phi, \qquad \text{... (2.10 } j)$$

$$\delta U = -\,\bar{\nu}, \qquad \text{... (2.10 } k)$$

$$\delta X^i - \Delta\xi^i = (\mu - \gamma + \bar{\gamma})\,\xi^i, \qquad \text{... (2.10 } l)$$

$$\bar{\delta}\xi^i - \delta\bar{\xi}^i = -\,2\bar{\alpha}\bar{\xi}^i - 2\alpha\xi^i, \qquad \text{... (2.10 } m)$$

$$\bar{\delta}\,(\phi\bar{\phi}) - \delta\psi = -\,4\bar{\alpha}\psi \qquad \text{... (2.10 } n)$$

and $\bar{\delta}\,(\phi\bar{\phi}) = 0$ [from (2.10 i) & (2.10 m)]. ... (2.10 o)

Eqs. (2.9 a) has the solution $\rho = (-\,r + \rho^0)^{-1}$, where ρ^o is the constant of integration and can be set equal to zero by (1.7 a), and a degree sign superscript indicates a function independent of r, and thus

$$\rho = -\frac{1}{r}. \qquad \text{... (2.11 } a)$$

The remaining radial equations have their solutions as :

$$\alpha = \alpha^0/r, \qquad \text{... (2.11 } b)$$

$$\beta = \beta^0/r, \qquad \text{... (2.11 } c)$$

$$\gamma = \gamma^0, \qquad \text{... (2.11 } d)$$

$$\mu = \mu^0/r, \quad \ldots (2.11\ e)$$

$$\nu = \nu^0, \quad \ldots (2.11\ f)$$

$$\psi = \psi^0/r, \quad \ldots (2.11\ g)$$

$$\phi = \phi^0/r, \quad \ldots (2.11\ h)$$

$$U = U^0 - (\gamma^0 + \bar{\gamma}^0)\, r, \quad \ldots (2.11\ i)$$

$$X^i = X^{0i} \quad \ldots (2.11\ j)$$

and $$\xi^i = x^{0i}/r \quad \ldots (2.11\ k)$$

From (1.7) it is possible to write $\xi^{0j} = a\,(\delta_3^j + \delta_4^j)$ and the function $a \equiv a\,(u, \zeta, \bar{\zeta})$ is real. Substituting the above results into the non-radial equation (2.10 *l*) and comparing the coefficients of like powers of $1/r$, we obtain

$$\mu^0 = U^0 \quad \ldots (2.12)$$

and $$\xi^{0j}\,(X^{03}_{\ \ j} + i\,X^{04}_{\ \ j}) \equiv a\ \nabla X^0 = 0, \quad \ldots (2.13)$$

where $\nabla \equiv \dfrac{\partial}{\partial x^3} + i\,\dfrac{\partial}{\partial x^4}$, $X^0 = X^{03} + iX^{04}$, and from (1.9) we have

$$X^{0i} = 0. \quad \ldots (2.14)$$

Eqs. (2.10 *l*) and (2.14) yield

$$2\gamma^0 = \frac{\partial}{\partial u} \log a. \quad \ldots (2.15)$$

The other non-radial equations can be worked out in the same manner as that of equation (2.10 *l*) and we have

$$(2.10\ k) \Rightarrow \bar{\nu}^0 = \frac{1}{2}\, a\ \nabla\left(\frac{\partial}{\partial u} \log a\bar{a}\right). \quad \ldots (2.16)$$

and $$(2.10\ m) \Rightarrow 2a^0 = \bar{a}\ \nabla \log a. \quad \ldots (2.17)$$

$$(2.10\ c) \Rightarrow \mu^0 = U^0 = 4\alpha^0\,\bar{\alpha}^0 = a\bar{a}\ \nabla \bar{\nabla} \log a \log \bar{a}. \quad \ldots (2.18)$$

$$(2.10\ a) \Rightarrow \psi^0 = \bar{a}\ \nabla \nu^0 + 2\alpha^0 \nu^0. \quad \ldots (2.19)$$

$$(2.10\ i) \Rightarrow a\ \nabla \psi^{0} = 4\,\bar{\alpha}^{0}\,\psi^{0}. \qquad \text{... (2.20)}$$

$$(2.10\ j) \Rightarrow a\ \nabla \phi^{0} = 2\alpha^{0}\phi^{0}. \qquad \text{... (2.21)}$$

$$(2.10\ o) \Rightarrow \bar{a}\ \bar{\nabla}\phi^{0}\,\bar{\phi}^{0} = 0 \Rightarrow \nabla\phi^{0}\,\bar{\phi}^{0} = \bar{\nabla}\phi^{0}\,\bar{\phi}^{0} = 0 \text{ and thus}$$

$$\phi^{0} = \phi^{0}\,(u). \qquad \text{... (2.22)}$$

From (1.8 *b*) it is possible to set $a = \bar{a}$ and thus

$$\alpha^{0} = \frac{1}{2}\,a\,A, \qquad \text{... (2.23)}$$

$$\mu^{0} = U^{0} = a^{2}B, \qquad \text{... (2.24)}$$

$$\gamma^{0} = \frac{1}{2}\,C \qquad \text{... (2.25)}$$

$$\nu^{0} = aD, \qquad \text{... (2.26)}$$

and $$\psi^{0} = a^{2}\ \nabla(1 + \log a)D, \qquad \text{... (2.27)}$$

where $A = \nabla \log a,\ B = \nabla^{2}\,(\log a)^{2},\ C = \frac{\partial}{\partial u}\log a$ and $D = \nabla C = \nabla\frac{\partial}{\partial u}\log a.$

The remaining non-radial equations are identically satisfied and hence the integration of all the quantities is completed.

III. The Solution and its Properties

From (1.5) and the simplifications of section II, the space-time representing the *PR* field is described by the metric

$$ds^{2} = E\,du^{2} + 2du\,dr + g_{ij}\,dx^{i}\,dx^{j}, \qquad \text{... (3.1)}$$

where $E = 2[a^{2}B - Cr]$ and $g^{ij} = (g_{ij})^{-1} = -\frac{2a^{2}}{r^{2}}\,\delta^{ij}$. Here the propagation vector l^{μ} of the two interacting fields is geodetic, shear-free, twist-free, non-expanding and hypersurface orthogonal and the tetrad $Z^{\mu}_{a} = \{l^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu}\}$ is parallelly propagated along l^{μ}. The interaction of gravitational radiation with electromagnetic radiation is described by equation (2.10 *n*). The solution (3.1) also exhibits singularities at $r = 0$ and $r \to \infty$.

It is interesting to note that the shear-free (i.e., $\sigma = 0$) and geodetic (i.e., $k = 0$) properties of the propagation vector l^{μ} are due to the interaction of two fields, while the parallel propagation of the tetrad Z^{μ}_{a} (i.e., $k = \varepsilon = \pi = 0$) and twist-free (i.e., $\rho - \bar{\rho} = 0$) property of l^{μ} have been achieved as a consequence of field equations, commutator relations (c.f. Section II).

In the absence of null electromagnetic field if γ, ν are zero and $\rho \neq \bar{\rho}$, then the solution (3.1) reduce to the space-time representing the spherical gravitational waves[18] and the metric obtained by Siklos[18], while the present solution becomes the well known Kerr twisting space-time[19] if $\rho \neq \bar{\rho}$, $\psi_i \neq 0$, $i = 2, 3, 4$. However, it may be noted that in these cases the choices of the tetrad vectors (1.4) and accordingly the coordinate transformations are different from those discussed here.

Using the null tetrad techniques[15] recently, Ahsan[20] has obtained the conditions under which an electromagnetic field (non-null and null) may admit a Maxwell collineation, Ricci collineation and motion, and for null electromagnetic fields it is seen that

$$\mathcal{L}_l F_{ij} = (\rho - 2\varepsilon)\,\phi l_{[\,i\, m_j]} + (\bar{\rho} - 2\bar{\varepsilon})\,\bar{\phi}\, l_{[\,i\, m_j]}$$

$$+ \frac{1}{2}\,\{(\tau l_i l_j + k l_i n_j - \rho l_i m_j - \sigma l_i \bar{m}_j)\,\phi$$

$$+ (\bar{\tau} l_i l_j + \bar{k}\, l_i n_j - \bar{\rho} l_i m_j - \bar{\sigma}\, l_i \bar{m}_j)\,\bar{\phi}\}$$

$$- \frac{1}{2}\,\{(\tau l_j\, l_i + k l_j n_i - \rho l_j m_i - \sigma l_j \bar{m}_i)\,\phi$$

$$+ (\bar{\tau} l_j l_i + \bar{k} l_j n_i - \bar{\rho} l_j m_i - \bar{\sigma} l_j \bar{m}_i)\,\bar{\phi}\},$$

$$\mathcal{L}_l R_{ij} = \chi\,(\rho + \bar{\rho})\, R_{ij}$$

and

$$\mathcal{L}_l g_{ij} = [(\gamma + \bar{\gamma})\, l_i l_j - \bar{\tau}\, l_i m_j - \tau l_i \bar{m}_j + (\varepsilon + \bar{\varepsilon})\, n_i l_j$$

$$- \bar{k} n_i m_j - k\, n_i \bar{m}_j - (\alpha + \bar{\beta}) m_i l_j + \bar{\sigma} m_i m_j + \rho m_i \bar{m}_j$$

$$- (\bar{\alpha} + \beta)\, \bar{m}_i\, l_j + \bar{\rho}\, \bar{m}_i m_j + \sigma \bar{m}_i\, \bar{m}_j + (\gamma + \bar{\gamma})\, l_j l_i$$

$$- \bar{\tau} l_j m_i - \tau l_j m_i + (\varepsilon + \bar{\varepsilon})\, n_j l_i - \bar{k} n_j m_i - k n_j \bar{m}_i$$

$$- (\alpha + \bar{\beta})\, m_j l_i + \bar{\sigma}\, m_j m_i + \rho\, m_j\, \bar{m}_i - (\bar{\alpha} + \beta)\, \bar{m}_j l_i + \bar{\rho}\, \bar{m}_j m_i + \sigma\, \bar{m}_j\, \bar{m}_i].$$

These equations suggest that the *PR* fields, discussed here, do not admit Ricci collineation and motion but they do admit Maxwell collineation along the propagation vector l^{μ}.

The Lanczos spin tensor[21] has again attracted the attention of several workers[18, 22]. Following[22], it can easily be seen that for (3.1), the Lanczos potential has the form

$$\Omega_1 = \frac{1}{6r},\ \Omega_6 = \frac{a^2 B}{6r},\ \Omega_7 = \frac{aD}{2}\ ,$$

$$\Omega_i = 0,\ i = 0,\ 2,\ 3,\ 4,\ 5,$$

which show that Ω_1 and Ω_6 are in radial direction while Ω_7 is independent of r.

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