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REAL HYPERSURFACES OF A COMPLEX HYPERBOLIC SPACE SATISFYING A POINTWISE NULLITY CONDITION

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In this paper, we give a classification of real hypersurfaces of a complex hyperbolic space $\mathbb{C}H^n$ satisfying a pointwise nullity condition for the structure vector field ξ i.e, $R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\}$ k is a function, and further we characterize a horosphere in $\mathbb{C}H^n$ by the condition $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$, that is, $k = 1$.

0. INTRODUCTION

Let $\mathbb{C}H^n = (\mathbb{C}H^n, J, \tilde{g})$ be an n-dimensional complex hyperbolic space with the Bergmann metric $\tilde{g}$ of constant holomorphic sectional curvature-4, and let M be an orientable real hypersurface of $\mathbb{C}H^n$ and N be a unit normal vector field on M. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kählerian structure $(J, \tilde{g})$ of $\mathbb{C}H^n$ (see section 1). Two typical examples of M is a geodesic hypersphere and a horosphere. J. Berndt[1] classified hypersurfaces with constant principal curvatures of $\mathbb{C}H^n$ under the condition that $-JN = \xi$ is a principal curvature vector. We denote by ∇ the Levi-Civita connection with respect to g. The curvature tensor field R on M is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ where X and Y are vector fields on M. It is known that $\mathbb{C}H^n$, $n \geq 3$ does not admit a real hypersurface with constant sectional curvature[5&7].

On the oth er hand, Tanno[9] defined for $k \in \mathbb{R}$ the k-nullity distribution $N(k)$ of a Riemannian manifold by $N(k) : p \rightarrow N_p(k) = \{z \in T_pM : R(x, y)z = k(g(y, z)x - g(x, z)y)$ for any $x, y \in T_pM\}$. If $T_pM = N_p(k)$ for any $p \in M$, then we see that M is of constant curvature k. In [4] we investigated a real hypersurface of $\mathbb{C}P^n$ whose structure vector field ξ satisfies a pointwise nullity condition, that is, $R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\}$, where k is a function on M. In the present paper, we consider a real hypersurface of $\mathbb{C}H^n$. In section 2, we give a classification of a real hypersurface M of $\mathbb{C}H^n$ which satisfies $R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\}$, where k is a function on M. In section 3, we investigate a real hypersurface M of $\mathbb{C}H^n$ which satisfies $R_\xi X = R(X, \xi)\xi = k$

$\{X-\eta(X)\xi\}$, where k is a function on M. Moreover, we prove that M satisfies $A^2\xi=\lambda A\xi$ and $\phi R_\xi = R_\xi \phi$ simultaneously if and only if M is locally congruent to a horosphere or a tube over totally geodesic $\mathbb{C}H^k$ $(0\le k\le n-1)$. In section 4, we determine real hypersurfaces which satisfy the property (*) ξ is a geodesic vector field and the Jacobi operator R_ξ is diagonalizable by a parallel orthogonal frame field along each trajectory of ξ and at the same time their eigenvalues are constant. We easily see that the property (*) is equivalent to the condition $R'_\xi=0$ where we denote $R'_X=(\nabla_X R)(\cdot,X)X$ for any vector field X. Also, in section 4 we show that $\mathbb{C}H^n$ does not admit a locally symmetric $(\nabla R=0)$ real hypersurface whose structure vector ξ is principal. In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

1. PRELIMINARIES

At first, we review the fundamental facts on a real hypersurface of $\mathbb{C}H^n$. Let M be a real hypersurface of $\mathbb{C}H^n$ and N be a unit normal vector field on M. By $\tilde\nabla$ we denote the Levi-Civita connection with respect to the Bergmann metric of $\mathbb{C}H^n$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde\nabla_X Y=\nabla_X Y+g(AX,Y)N,\ \tilde\nabla_X N=-AX$$

for any tangent vector fields X **and** Y **on** M, where g denotes the Riemannian metric of M induced from $\tilde g$. An eigenvector (resp. eigenvalue) of the shape operator A is called a principal curvature vector (resp. principal curvature). For any vector field X tangent to M, we put

$$JX=\phi X+\eta(X)N,\ JN=-\xi. \qquad \text{... (1.1)}$$

Then we may see that the structure (ϕ,ξ,η,g) is an almost contact metric structure on M, that is, we have

$$\left.\begin{array}{l}\phi^2X=-X+\eta(X)\xi,\ \eta(\xi)=1,\\ g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)\end{array}\right] \qquad \text{... (1.2)}$$

From (1.2), we get

$$\phi\xi=0,\ \eta\circ\phi=0,\ \eta(X)=g(X,\xi). \qquad \text{... (1.3)}$$

From the fact $\tilde\nabla J=0$ and (1.1), making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi, \qquad \text{... (1.4)}$$

$$\nabla_X\xi=\phi AX. \qquad \text{... (1.5)}$$

Since the ambient space is of constant holomorphic sectional curature-4, we have the following Gauss and Codazzi equations :

$$R(X, Y)Z = - \{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY. \quad \ldots (1.6)$$

$$(\nabla_X A)Y - (\Delta_Y A)X = -\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}. \quad \ldots (1.7)$$

From (1.6), using (1.2), (1.3), then the Ricci tensor S is given by

$$SX = -(2n+1)X + 3\eta(X)\xi + hAX - A^2X, \quad \ldots (1.8)$$

where h = the trace of A. We recall the following

Proposition 1[6] — If ξ is a principal curvature vector field, then the corresponding principal curvature α_1 is constant.

A real hypersurface M of $\mathbb{C}H^n$ is said to be *totally* η*-umbilical,* i.e., M satisfies $AX = aX + b\eta(X)\xi$ for any tangent vector field X on M, where a and b are constants on M (cf. [10]).

Theorem 1[7,8] — *Let M be a real hypersurface of* $\mathbb{C}H^n$. *Then M is totally* η*-umbilical if and only if M is locally congruent to a horosphere, a tube of radius* $r \in \mathbb{R}_+$ *over totally geodesic* $\mathbb{C}H^0$ *(that is, geodesic hypersphere) or a tube of radius* $r \in \mathbb{R}_+$ *over totally geodesic* $\mathbb{C}H^{n-1}$.

Furthermore, we prepare the following :

Theorem 2[8] — *Let M be a real hypersurface of* $\mathbb{C}H^n$. *Then the followings are equivalent :*

(i) M is locally congruent to a horosphere or a tube of radius $r \in \mathbb{R}_+$ *over totally geodesic* $\mathbb{C}H^k$ $(0 \le k \le n-1)$;

(ii) $\phi A = A\phi$.

We define a vector field U on M by $U = \nabla_\xi \xi$ and denote $\alpha_m = \eta(A^m \xi)$. Then from (1.2) and (1.5) we easily observe that

$$\text{and} \quad \left.\begin{array}{l} g(U, \xi) = 0,\ g(U, A\xi) = 0, \\ \|U\|^2 = g(U, U) = \alpha_2 - \alpha_1^2. \end{array}\right] \quad \ldots (1.10)$$

From (1.2), (1.5) and (1.10) we have at once

Lemma 1 — Let M be a real hypersurface of $\mathbb{C}H^n$. Then ξ is a principal curvature vector field if and only if M satisfies $\alpha_2 - \alpha_1^2 = 0$.

Now, we recall that[9] the k-nullity distribution of a Riemannian manifold, for a real number k, is a distribution

$$N(k) : p \to N_p(k) = \{z \in T_pM : R(x, y)z = k(g(y, z)x - g(x, z)y) \text{ for any } x, y \in T_pM\}.$$

If $T_pM = N_p(k)$ for any point $p \in M$, then we see that M is of constant curvature k. In section 2, we consider a pointwise nullity condition for the structure vector field ξ.

2. Real Hypersurfaces Satisfying a Pointwise Nullity Condition

In this section, we give a classification of a real hypersurface of $\mathbb{C} H^n$ whose structure vector field ξ satisfying

$$R(X,Y)\xi = k\ \{\eta(Y)X - \eta(X)Y\} \qquad \text{... (2.1)}$$

for a function k, where X, Y are vector fields tangent to M. Thus from Theorem 1 and (1.6) we have

Proposition 2 — Let M be a horosphere, a geodesic hypersphere or a tube over totally geodesic $\mathbb{C} H^{n-1}$. Then M satisfies (2.1) for a constant k.

In order to prove the converse of the above Proposition 2, we prove first

Lemma 2 — Let M be a real hypersurface of $\mathbb{C} H^n$. If M satisfies (2.1), then ξ is principal.

PROOF : From (1.6) and (2.1) we have

$$(k+1)\ \{\eta(Y)X - \eta(X)Y\} = \eta(AY)AX - \eta(AX)AY \qquad \text{... (2.2)}$$

for any vector field X and Y. Putting $X = \xi$ and $Y = U$ in (2.2), then together with (1.10) we get

$$\alpha_1 AU = (k+1)U. \qquad \text{... (2.3)}$$

We put again $X = A\xi$ and $Y = U$ is (2.2), we get then

$$\alpha_2 AU = \alpha_1 (k+1)\, U. \qquad \text{... (2.4)}$$

From (2.3) and (2.4) we obtain

$$(k+1)(\alpha_2 - \alpha_1^2)\, U = 0. \qquad \text{... (2.5)}$$

We set $P = \{p \in M : k(p) \neq -1\}$ and $Q = \{p \in M : k(p) = -1\}$. Then we see that $M = P \cup Q$. Here, we shall show that Q is empty. Suppose that Q is non-empty, then from (2.2) it follows that

$$\eta(AY)AX - \eta(AX)AY = 0 \qquad \text{... (2.6)}$$

for any vector field X and Y on Q. We put $X = \xi$ in (2.6), then we get

$$\alpha_1 AY = \eta(AY)A\xi \qquad \text{... (2.7)}$$

for any vector field Y on Q. If there exists a point $p \in Q$ such that $\alpha_1(p) \neq 0$, then from (2.7) we see that the rank of the shape operator A at p, which is called the type number at p, is at most 1. In this case, we see that the point p is a geodesic point (*f*, [*n*]). Thus, it follows that $\alpha_1 = 0$ on Q, and hence we get $\alpha_2 = 0$ on Q. Again applying Lemma 1 and Proposition 1, we see that $A\xi = 0$ on M. But we know that $A\xi \neq 0$ (cf. [1]). Thus, we see that Q must be empty, that is, $M = P$. Therefore, from (2.5) and lemma 1 we see that ξ is principal on M. (Q.E.D.)

Theorem 3 — *Under the same assumption as that of Lemma 2, then M is locally congruent to a horosphere, a geodesic hypersphere, or a tube over* $\mathbb{C}H^{n-1}$.

PROOF : It follows from $A\xi = \alpha_1 \xi$ and (2.2) that

$$(k+1)\ \{\eta(Y)X - \eta(X)Y\} = \alpha_1\ \{\eta(Y)AX - \eta(X)AY\}. \qquad \text{... (2.8)}$$

We know that α_1 is constant (Proposition 1) and by the result of [1] know that α_1 can not be zero. If we assume $Y = \xi, X \perp \xi$ in (2.8), then we get

$$AX = (k+1)/\alpha_1 X \qquad \text{... (2.9)}$$

for any vector field X orthogonal to ξ. Hence from (2.9) we see that M has at most two distinct principal curvatures α_1 and $\lambda = (k+1)/\alpha_1$ with multiplicities 1 and $2n-2$, respectively. So, we see that the shape operator A is represented by

$$A = (k+1)/\alpha_1 I + (\alpha_1 - (k+1)/\alpha_1)\,\eta \otimes \xi. \qquad \text{... (2.10)}$$

We now show that k is constant. If we differentiate A covariantly with respect to any tangent vector X, then since α_1 is constant (Proposition 1) we have

$$(\nabla_X A)Y = (Xk)/\alpha_1\{Y - \eta(Y)\xi\} + (\alpha_1 - (k+1)/\alpha_1)$$
$$\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\}. \qquad \text{... (2.11)}$$

But using the Codazzi eqs. (1.7) and (2.10) we have

$$(Xk)/\alpha_1\ \phi X - (\phi Xk)/\alpha_1 X = 2g(X, X)\xi \qquad \text{... (2.12)}$$

for any $X \perp \xi$. From (2.12), we get $Xk = 0$, that is, k is constant in every direction orthogonal to ξ. Also, if we put $X = \xi$ in (2.11) and by using (1.7) again we obtain

$$(\xi k)/\alpha_1 Y - (\alpha_1 - (k+1)/\alpha_1)(k+1)/\alpha_1\ \phi Y = -\phi Y$$

for any $Y \perp \xi$, which yields $\xi k = 0$, and thus we see that k is constant. Thus, together with (2.10) we see that M is totally η-umbilical. Thus by Theorem 1 we see that M is locally congruent to a horosphere, a geodesic hypersphere or a tube over $\mathbb{C}H^{n-1}$. (Q.E.D.)

In particular, it is known that a horosphere is totally η-umbilical and is determined by $A = I + \eta \otimes \xi$ (see [1]). Thus, we have

Corollary 1 — Let M be a real hypersurface of $\mathbb{C}H^n$. Then M satisfies $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$ for any vector fields X, Y on M if and only if M is locally congruent to a horosphere.

We denote $h^{(m)}$ = trace A^m, then in particular $h^{(1)} = h$ in (1.8). We also prove

Proposition 3 — Let M be a real hypersurface of $\mathbb{C}H^n$. Then M always satisfies

$$H_1^2 \leq 2(n-1)H_2,$$

where we put $H_m = h^{(m)}\alpha_m - \alpha_{2m}$. If the equality holds, then ξ is principal. Moreover, M is locally congruent to a horosphere, a geodesic sphere or a tube over $\mathbb{C}H^{n-1}$.

PROOF : We put

$$T(X, Y) = R(X, Y)\xi - k\{\eta(Y)X - \eta(X)Y\}$$

for any vector fields X and Y on M, where k is a function. Then T is a (1, 2)-tensor field on M. We calculate $\|T\|^2$, then we have

$$\|T\|^2 = \sum_{i,j} g(R(e_i, e_j)\xi - k\{\eta(e_j)e_i - \eta(e_i)e_j\},\ R(e_i, e_j)\xi - k\{\eta(e_j)e_i - \eta(e_i)e_j\})$$

$$= \|R(\cdot,\cdot)\xi\|^2 - 4k\eta(S\xi) + 4(n-1)k^2, \qquad \text{... (2.13)}$$

where $\{e_i\}$ $(i = 1, 2, .., 2n-1)$ is an orthonormal basis of the tangent space. From (1.6) and (1.8) a direct calculation yields

$$\|R(\cdot,\cdot)\xi\|^2 = 4(n-1) - 4H_1 + 2H_2 \qquad \text{... (2.14)}$$

and

$$\eta(S\xi) = -(2n-2) + H_1. \qquad \text{... (2.15)}$$

From (2.13), (2.14) and (2.15) we have

$$\|T\|^2 = 4(n-1)(1+k)^2 - 4H_1(1+k) + 2H_2 \geq 0. \qquad \text{... (2.16)}$$

Since (2.16) holds for any k at any (fixed) point on M, we see that

$$H_1^2 \leq 2(n-1)H_2. \qquad \text{... (2.17)}$$

Further we see that the equality holds in (2.17) if and only if $\|T\|^2 = 0$. Thus by using Theorem 3, we have our conclusion. (Q.E.D.)

3. REAL HYPERSURFACES OF $\mathbb{C}H^n$ SATISFYING $R_\xi = k(I - \eta\otimes\xi)$

For each point $p \in M$ and each unit tangent vector $X \in T_pM$, we define a self-adjoint operator R_X of T_pM by $R_X = R(\cdot, X)X$. We call R_X Jacobi operator with respect to X. In this section, we prove

Theorem 4 — *Let M be a real hypersurface of $\mathbb{C}H^n$ whose structure vector ξ is principal. Suppose that M satisfies*

$$R_\xi = k(I - \eta\otimes\xi), \qquad \text{... (3.1)}$$

where k is a function on M. Then M is locally congruent to a horosphere, a geodesic hypersphere or a tube over $\mathbb{C}H^{n-1}$.

PROOF : From (1.6) it follows that

$$R_{\xi} X = R(X, \xi)\, \xi = -X + \eta(X)\, \xi + \alpha_1 AX - \eta\,(AX)\, A\xi \qquad \text{... (3.2)}$$

for any vector field X on M. From (3.1) and (3.2) we get

$$\alpha_1 AX = (k+1)\,(X - \eta\,(X)\,\xi) + \eta\,(AX)\, A\xi \qquad \text{... (3.3)}$$

for any vector field X on M. Since $A\xi = \alpha_1\, \xi$ and α_1 is non-zero constant, from (3.3) we get

$$AX = (k+1)/\alpha_1\, X$$

for any vector field X orthogonal to ξ. Thus by the same arguments as in the proof of Theorem 3, we see that M is locally congruent to a horosphere, a geodesic hypersphere or a tube over $\mathbb{C}\, H^{n-1}$. (Q.E.D.)

Remark 1 : From Theorem 4, we see that the normal Jacobi vector field along each geodesic trajectory of ξ on a horosphere, geodesic hypersphere, or a tube over $\mathbb{C}\, H^{n-1}$ satisfies the space form type Jacobi equation, i.e., $Y'' + kY = 0$ where k is a constant and $'$ denotes covariant derivative along a geodesic trajectory of ξ.

Here, we consider the case that ξ is not principal and M satisfies (3.1). Then we may assume that

$$A\xi = \alpha_1\, \xi + \mu W, \qquad \text{... (3.4)}$$

where W is unit and orthogonal to ξ, $\mu \neq 0$.Then from (3.2) and (3.4) we get

$$AW = \mu\xi + (\mu^2 + k + 1)/\alpha_1\, W. \qquad \text{... (3.5)}$$

Also, from (3.2) we get

$$AZ = (k+1)/\alpha_1\, Z, \qquad \text{... (3.6)}$$

where Z is unit and orthogonal to ξ and W.

Now we prove that $\alpha_1 \neq 0$. If α_1 vanishes identically on M, then from (3.2) we get

$$(k+1)\,(X - \eta(X\,\xi) + \eta(AX)\, A\xi = 0 \qquad \text{... (3.7)}$$

for any vector field X on M. Putting $X = U$ in (3.7), then we obtain

$$(k+1)\, U = 0. \qquad \text{... (3.8)}$$

Set $\Omega_1 = \{p \in M : k(p) = -1\}$ and $\Omega_2 = \{p \in M : k(p) \neq -1\}$. Then $M = \Omega_1 \cup \Omega_2$. If $M = \Omega_1$ (Ω_2 is empty), then from (3.7) we find $\alpha_2 = 0$ on M, and hence by applying Lemma 1 we have $A\xi = 0$ on M. If $M = \Omega_2$ (Ω_1 is empty), then from (3.8) and Lemma 1 we see that $A\xi = \alpha_1\, \xi$ on M. Or, in case that both Ω_1 and Ω_2 are non-empty, by Proposition 1 and the continuity of α_1 yield that ξ is a principal curvature vector field on M. Hence, all the cases yield a contradiction. Thus $\alpha_1 \neq 0$ on M.

Also, we show that $k \neq -1$. The mark of A at a point p in M is called a type number and is denoted by $t(p)$. Let M be a real hypersurface of $\mathbb{C} H^n$ which satisfies $R_\xi X = -X + \eta(X)\,\xi$, i.e., $k = -1$. Then from (3.3) it follows that

$$\alpha_1 AX = \eta(AX)\, A\xi \qquad \text{... (3.9)}$$

for any vector field X on M. If there exist a point p in M such that $\alpha_1(p) \neq 0$, then (3.9) implies that the type number $t(p)$ at p is at most 1. It is however seen (cf. [10]) that the point p is geodesic. So it is contradictory to the assumption that $\alpha_1(p) \neq 0$. Thus $\alpha_1 = 0$ on M, which is impossible. Therefore, we see that $k \neq -1$. Thus, from (3.4), (3.5) and (3.6) it follows that the shape operator A is written as

$$\left.\begin{aligned} &A\xi = \alpha_1\, \xi + \mu\, W\, (\alpha_1 \neq 0, \mu \neq 0), \\ &AW = \mu\xi + \nu\, W, \\ &AZ = (k+1)/\alpha_1\, Z,\, k+1 = \alpha_1\, \nu - \mu^2 \end{aligned}\right] \qquad \text{... (3.10)}$$

for any $Z \perp \xi, W$, where W is a unit vector field orthogonal to ξ, α_1, μ and ν are functions on M.

We see that all the ones appeared in Theorem 4 satisfies $\phi R_\xi = R_\xi\, \phi$. Now, we prove

Theorem 5 — Let M be a real hypersurface of $\mathbb{C} H^n$. *The structure tensor* ϕ *commutes with the jacobi operator* R_ξ *and* $A\xi$ *is a principal curvature vector field on M. Then* ξ *is principal on M. Furthermore M is locally congruent to a horosphere, a geodesic sphere or a tube over totally geodesic* $\mathbb{C} H^k$ $(1 \leq k \leq n-1)$.

PROOF : Assume that $\phi R_\xi = R_\xi\, \phi$ and $A^2\, \xi = \lambda A \xi$. From (1.6) we get

$$\left.\begin{aligned} &R_\xi(\phi X) = -\,\phi X + \alpha_1 A\, \phi X + g(X, U)\, A\xi, \\ &\phi(R_\xi X) = -\,\phi X + \alpha_1\, \phi AX - g\,(AX, \xi) U. \end{aligned}\right] \qquad \text{... (3.11)}$$

From (3.11) and the assmption $\phi R_\xi = R_\xi\, \phi$, we find

$$\alpha_1\, (\phi A - A\phi)\, X = g(X, U)\, A\xi + g(AX, \xi)\, U. \qquad \text{... (3.12)}$$

First, we prove that ξ is principal on M. We put $X = A\xi$ in (3.12) and using the another assumption $A^2\, \xi = \lambda A \xi$, then we get $\alpha_1 AU = (\alpha_1 \lambda - \alpha_2) U$, and hence we have

$$\alpha_1 AU = 0, \qquad \text{... (3.13)}$$

because $\alpha_2 = \alpha_1\, \lambda$. If there exists a point $p \in M$ such that $\alpha_1(p) = 0$, then we see that $\alpha_2(p) = 0$, and hence by using Lemma 1, we conclude that $A\xi = 0$ at p. So, from now we discuss where α_1 has not zero. Then from (3.13), it follows that

$$AU = 0. \qquad \text{... (3.14)}$$

With (3.14) we easily see that

$$g((\nabla_X A)\,\xi,\xi) = d\alpha_1\,(X),$$

where d denotes the exterior differential. Since $U = \phi A\xi$, from (1.4), (1.7) and (3.14) we have

$$\nabla_\xi U = \alpha_1 A\xi - \alpha_2\xi - \phi\,\nabla\alpha_1, \qquad \text{... (3.15)}$$

where $\nabla\alpha_1$ denotes the italic gradient vector of α_1. Differentiating (3.14) covariantly, then by using (1.7) and (3.15) we have

$$(\nabla_U A)\,\xi = \phi U - \alpha_1 A^2\xi + \alpha_2 A\xi - A\,\phi\,\nabla\alpha_1. \qquad \text{... (3.16)}$$

Also, differentiating $A^2\xi = \lambda A\xi$ covariantly along M, then together with (1.5) we have

$$g(A\xi,(\nabla_X A)Y) + g((\nabla_X A)\,\xi, AY) + g(\phi AX, A^2 Y)$$

$$= d\lambda(X)\,g(A\xi, Y) + \lambda g\,((\nabla_X A)\,\xi, Y) + \lambda g\,(\phi AX, AY).$$

From (1.7) and (3.17) we have

$$-\ \eta(X)\,g(A\xi,\phi Y) + \eta(Y)\,g(A\xi,\phi X) + 2\alpha_1\,g(\phi X, Y)$$

$$+\ g(\nabla_X A)\,\xi, AY) - g((\nabla_Y A)\,\xi, AX) + g(\phi AX, A^2 Y) - g(\phi AY, A^2 X)$$

$$= d\lambda(X)\,g(A\xi, Y) - d\lambda(Y)\,g(A\xi, X) + \lambda g\,((\nabla_X A)\,\xi, Y) - \lambda g\,((\nabla_Y A)\,, X)$$

$$+\ 2\lambda g\,(\phi AX, AY)$$

for any vector fields X and Y on M. We put $X = U$ and making use of (1.7), (3.14) and (3.16), then we have

$$g((\nabla_U A)\,\xi, AY) = 2(\lambda - \alpha_1)\,g\,(\phi U, Y) + \eta(Y)\,g(U, U) + d\lambda(U)\,g(A\xi, Y). \qquad \text{... (3.18)}$$

Thus, from (3.16) and (3.18) we obtain

$$2(\lambda - \alpha_1)g(\phi U, Y) + \eta(Y)\,g(U, U) + d\lambda(U)g(A\xi, Y)$$

$$= g(\phi U, AY) - \alpha_1 g(A^2\xi, AY) + \alpha_2 g(A\xi, AY) + d\alpha_1\,(\phi A^2 Y). \qquad \text{... (3.19)}$$

Putting $Y = \xi$ in (3.19), then we get

$$\alpha_1 d(\lambda)(U) - \lambda d(\alpha_1)(U) = -\,2(\alpha_2 - \alpha_1^2).$$

Also, we put $Y = A\xi$ in (3.19), we get

$$\lambda\{\alpha_1\,d(\lambda)\,(U) - \lambda d(\alpha_1)(U)\} = (\alpha_2 - \alpha_1^2)\,(\lambda - 3\alpha_1).$$

Thus, we have $\alpha_2 - \alpha_1^2 = \alpha_1(\lambda - \alpha_1) = 0$, from which using Lemma 1 we see that $A\xi = \alpha_1 \xi$ on M, where α_1 is a non-zero constant.

From (3.12) and Lemma 1, we see that

$$\alpha_1 (\phi A - A\phi)X = 0.$$

Since α_1 is non-zero constant, by Theorem 2 we have our assertions. (Q.E.D)

Remark 2 : If a real hypersurface of $\mathbb{C} H^n$ satisfies (3.10) and $A^2\xi = \lambda A \xi$ at the same time, then we can see that $\alpha_1 \nu - \mu^2 = k + 1 = 0$, which can not occur.

4. Real Hypersurfaces of $\mathbb{C} H^n$ Satisfying $R'_X = 0$

For each point $p \in M$ and each unit tangent vector $X \in T_pM$, we define R'_X by $R'_X = (\nabla_X R)(\cdot, X)X$. Then in particular supposing that the structure vector field ξ of M is a geodesic vector field, it is easily seen that $R'_\xi = 0$ on M if and only if the Jacobi operator R_ξ is diagonalizable by a parallel orthonormal frame field along each trajectory of ξ and at the same time their eigenvalues are constant along each trajectory of ξ (cf. [2] or [3]). Now we prove

Proposition 4 — Let M be a real hypersurface of $\mathbb{C} H^n$. Suppose that ξ is a geodesic vector field on M and M satisfies $R'_\xi = 0$. Then ξ is principal curvature vector field on M. Furthermore, M is locally congruent to a horosphere, a geodesic sphere, or a tube over totally geodesic $\mathbb{C} H^k$ $(1 < k \leq n-1)$

Proof : Assume that ξ is a geodesic vector field on M. Then by Lemma 1, we immediately see that $A\xi = \alpha_1 \xi$. Differentiating covariantly this, and then by using Proposition 1 and (1.5) we have

$$(\nabla_X A)\xi = \alpha_1 \phi AX - A\phi AX,$$

and further by using (1.7) we obtain

$$(\nabla_\xi A)X = -\phi X + \alpha_1 \phi AX - A\phi AX$$

for any vector field X on M. Taking the skew-symmetric part of $\nabla_\xi A$, then we have

$$2A\,\phi AX + 2\phi X = \alpha_1(\phi A + A\phi)X \qquad \text{... (4.1)}$$

for any vector field X on M. From (1.6), taking account of (1.4), (1.7) and Proposition 1, we get

$$R'_\xi Y = (\nabla_\xi R)(Y, \xi)\xi = \alpha_1 (\nabla_\xi A) Y$$

$$= \alpha_1 (\alpha_1 \phi AY - A\phi AY - \phi Y)$$

for any vector field Y on M. Thus from the hypothesis we get

$$\alpha_1(\alpha_1\phi A - A\phi A - \phi)\, Y = 0. \qquad \text{... (4.2)}$$

If we assume $AY = \lambda Y$ for some Y orthogonal to ξ, then from (4.1) it follows that

$$(2\lambda - \alpha_1)\, A\phi Y = (\alpha_1\lambda - 2)\phi Y.$$

Thus, from (4.2) we have

$$\alpha_1(\lambda^2 - \alpha_1\lambda + 1) = 0,$$

where $2\lambda \neq \alpha_1$. We see that $\lambda^2 - \alpha_1\lambda + 1 = 0$ implies $\lambda(2\lambda - \alpha_1) = \alpha_1\lambda - 2$. If $2\lambda = \alpha_1$, then $\lambda = 1$ and $\alpha_1 = 2$, that is, M is locally congruent to horosphere. From these we see that M satisfies $\phi A = A\phi$, and hence by Theorem 2 we have our assertions. (Q.E.D.)

Proposition 5 — There does not exist a real hypersurface of $\mathbb{C}H^n$ whose structure vector ξ is principal and satisfying $R'_V = 0$ for any vector V orthogonal to ξ.

PROOF : From (1.6), taking account of (1.4), we get

$$\begin{aligned}(\nabla_v R)(Y, V)V = 3\{\eta(Y)\, g(AV, V)\phi V - g(\phi Y, V)\, g(AV, V)\xi\}\\ + g((\nabla_V A)(V, V)AY + g(AV, V)(\nabla_V A)Y\\ - g((\nabla_V A)Y, V)AV - g(AY, V)(\nabla_V A)V \qquad \text{... (4.3)}\end{aligned}$$

for any vector field Y on M and any vector field V orthogonal to ξ. Assume that $A\xi = \alpha_1\xi$ and suppose that M satisfies $R'_V = (\nabla_V R)(\cdot, V)V = 0$ for any vector field V orthogonal to ξ. Then of course $R'_V = (\nabla_V R)(\xi, V)\, V = 0$ and from (4.3)

$$\begin{aligned}3g(AV, V)\,\phi V + \alpha_1 g\,((\nabla_V A)\, V, V)\xi + g(AV, V)\,(\alpha_1\phi AV - A\,\phi AV)\\ - \alpha_1 g(\phi AV, V)\, AV + g(A\phi AV, V)\, AV = 0 \qquad \text{... (4.4)}\end{aligned}$$

for any vector field V orthogonal to ξ. From (4.4) we easily see that $\alpha_1 g\,((\nabla_V A)\, V, V)\xi = 0$ and have

$$3g(AV, V)\phi V + g(AV, V)\,(\alpha_1\phi AV - A\,\phi AV) - \alpha_1 g(\phi AV, v)\, AV + g(A\phi AV, V)\, AV = 0 \qquad \text{... (4.5)}$$

Assume $AV = \lambda V$. Then from (4.5) and Proposition 2 we have

$$\lambda\{\alpha_1\lambda^2 + (8 - \alpha_1^2)\,\lambda - 3\alpha_1\} = 0, \qquad \text{... (4.6)}$$

where $2\lambda \neq \alpha_1$. From (4.6) and Proposition 1 we see that M has constant principal curvatures. But, by virtue of the table in [1] we see that there does not exist M whose structure vector ξ is principal and satisfying (4.6). If M is a horosphere ($2\lambda = \alpha_1$), from (4.4) we get $\phi V = 0$, which is impossible. Thus, we have our assertions. (Q.E.D.)

From the above proposition 5, it follows

Corollary 2 — There does not exist a locally symmetric real hypersurface of $\mathbb{C} H^n$ whose structure vector ξ is principal.

Remark 3 : The result in [5] says that there does not exist a real hypersurface M with the parallel Ricci tensor in $\mathbb{C} H^n$, $n \geq 3$.

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