Indian J. pure appl. Math., **31**(3) : 277-286, March 2000

APPLICATIONS OF THE GROETSCH THEOREM

DRAGAN S. DJORDJEVIĆ AND PREDRAG S. STANIMIROVIĆ

University of Nis, Faculty of Philosophy, Department of Mathematics, Cirila I Metodija 2, 18000 Nis, Yugoslavia

(*Received* 24 *December* 1998; *Accepted* 21 *May* 1999)

In this paper, we investigate general representations of various classes of generalized inverses of bounded operators over Hilbert spaces, based on the full-rank factorization of operators. Using these general representations we introduce a generalization of the Groetsch representation theorem for the Moore-Penrose inverse. As corollaries, we derive a few iterative methods for computing reflexive g-inverses. In a particular case we get the main result from [9]. The present method is compared with [6].

Key Words : Groetsch Theorem; Banach Spaces; Drain Inverse; Hilbert Spaces; Moore-Penrose Inverse; Weierstress Approximation; Reflexive Generalized Inverses

1. INTRODUCTION

Let χ_1 and χ_2 denote arbitrary Banach spaces and $B(\chi_1, \chi_2)$ denote the set of all bounded operators from χ_1 into χ_2. For an arbitrary operator $A \in B(\chi_1, \chi_2)$, we use $\mathcal{N}(A)$ to denote its kernel, and $\mathcal{R}(A)$ to denote the image. An operator $A \in B(\chi_1, \chi_2)$ is g-invertible, provided that there exists some $X \in B(\chi_2, \chi_1)$, such that $AXA = A$. In this case X is called a g-inverse of A. If X satisfies both of the equations $AXA = A$ and $XAX = X$, then X is called a reflexive g-inverse of A. It is well known that an operator $A \in B(\chi_1, \chi_2)$ has a g-inverse if and only if $\mathcal{R}(A)$ is closed, and $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are complemented subspaces of χ_1 and χ_2 respectively. An arbitrary right inverse and an arbitrary left inverse of A are denoted by A_r^{-1} and A_l^{-1}, respectively.

We say that $A \in B(\chi)$ has the Drazin inverse, if there exists an operator $A^D \in B(\chi)$, such that A^D satisfies eq. (2) and the equations

$$(1^k)\ A^{k+1} A^D = A^k, \quad (5)\ A^D A = AA^D,$$

for some non-negative integer k. Let us mention that the Drazin inverse, if it exists, is unique. The smallest k in the previous definition is called the index of A and denoted by ind (A). In the case ind (A) = 1 the Drazin inverse is known as the group inverse of A, denoted by $A^{\#}$.

The full rank factorization of matrices is well-known and frequently used in representations of pseudoinverses[1,7,8,10]. The following analogy of the full rank factorization for matrices is established in [2], [3] :

Let $A \in B(\chi_1, \chi_2)$. If there exist a Banach space χ_3 and operators $Q \in B(X_1, X_3)$ and $P \in B(\chi_3, \chi_2)$, such that P is left invertible, Q is right invertible and

$$A = PQ, \qquad \text{... (1.1)}$$

then we say that (1.1) is the full-rank decomposition of A.

It is well known that an operator $A \in B(\chi_1, \chi_2)$ has the full-rank decomposition, if and only if A is g-invertible. In this case χ_3 is isomorphic to $\mathcal{R}(A)$, and $\mathcal{R}(A) = \mathcal{R}(P)$[3].

In the case when $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, it is well known that an operator $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has a g-inverse if and only if $\mathcal{R}(A)$ is closed. We consider the following equations in X :

$$\text{(1) } AXA = A, \text{ (2) } XAX = X, \text{ (3) } (AX)^* = AX, \text{ (4) } (XA)^* = XA.$$

For a subset $\mathcal{S}$ of the set $\{1, 2, 3, 4\}$, the set of operators obeying the conditions contained in $\mathcal{S}$ denoted by $A\{\mathcal{S}\}$. An operator in $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and is denoted by $A^{(\mathcal{S})}$. If $\mathcal{R}(A)$ is closed, the set A $\{1, 2, 3, 4\}$ consists of single element, the Moore-Pensrose inverse of A, denoted by $A^{\dagger}$.

A basic tool used in this paper is the following general representation theorem for the Moore-Penrose inverse of a bounded linear operator[3-5] :

Theorem 1.1 — *Let* $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ *has closed range. Then [5, p. 45]*

$$T^{\dagger} = \tilde{T}^{-1} T^*, \text{ where } \tilde{T} = T^* T_{|\mathcal{R}(T^*)}. \qquad \text{... (1.2)}$$

Moreover, if Ω is an open set with $\sigma(\tilde{T}) \subset \Omega \subset (0, \infty)$ and $\{S_{\beta}(x)\}^{\beta}$ is a family of continuous real valued functions on Ω, with $\lim_{\beta} S_{\beta}(x) = \frac{1}{x}$ uniformly on $\sigma(\tilde{T})$, then [3, p. 42], [4], [5, p. 57]

$$T^{\dagger} = \lim_{\beta} \; S_{\beta}(\tilde{T})\, T^*, \qquad \text{... (1.3)}$$

where the convergence is in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$. Further more,

$$\| S_{\beta}(\tilde{T})\, T^* - T^{\dagger} \| \leq \sup_{x \in \sigma(\tilde{T})} | x\, S_{\beta}(x) - 1 | \cdot \| T^{\dagger} \|.$$

We investigate general representations of bounded operators on Hilbert spaces, based on the full-rank factorization (1.1). These representations are extensions of known results from [2], [7], [8] and [10].

Using these general representations together with the Groetch representation theorem for the Moore-Penrose inverse of a bounded operator on Hilbert spaces, we introduce representations for various subsets of the set of all reflexive g-inverse of a bounded operator. Using this extension of the Groetch representation theorem, as particular cases, we derive a few iterative methods for computing g-inverses. As a partial result we get an improvement of the hyper-power iterative method, which is investigated in [9] for operators acting on finite dimensional complex Hilbert spaces. This method is not known for matrices before.

2. RESULTS

Firstly, we state the following general representations based on the full rank factorization of operators. These representations are known for matrices (see [7], [8] and [10]). For bounded operators between Hilbert spaces it is known a representation of the Moore-Penrose inverse, introduced in [2].

Lemma 2.1 — Let $A = PQ$ be a full-rank decomposition of $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ according to (1.1). Then :

(*a*) $X \in A\ \{1, 2\}$ if and only if there exist operators $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$ and $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, such that QW_1 and W_2P are invertible in $B(\mathcal{H}_3)$. In such a case, X possesses the following general representation

$$X = Q_r^{-1} P_l^{-1},\quad Q_r^{-1} = W_1 (QW_1)^{-1},\quad P_l^{-1} = (W_2P)^{-1} W_2. \qquad \dots (2.1)$$

(*b*) $X \in A\ \{1, 2, 3\}$ if and only if there exists an operator $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, such that QW_1 is invertible in $B(\mathcal{H}_3)$. In the case when it exists, a general representation for X is as follows

$$X = W_1 (QW_1)^{-1} (P^* P)^{-1} P^*. \qquad \dots (2.2)$$

(*c*) $X \in A\ \{1, 2, 4\}$ if and only if there exists an operator $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, such that $W_2 P$ is invertible in $B(\mathcal{H}_3)$. In this case

$$X = Q^*(QQ^*)^{-1} (W_2P)^{-1}W_2.$$

(*d*) $A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-} (P^*P)^{-1} P^* = Q^* (P^* AQ^*)^{-1} P^*$ [2].

(*e*) Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $X : \mathcal{H}_2 \to \mathcal{H}_1$. Then $X \in A\ \{2\}$ if and only if there exist operators

$$C \in B(\mathcal{H}_4, \mathcal{H}_1), D \in B(\mathcal{H}_2, \mathcal{H}_3), W_1 \in B(\mathcal{H}_5, \mathcal{H}_4), W_2 \in B(\mathcal{H}_3, \mathcal{H}_5),$$

such that DAC is g-invertible and $W_2 DACW_1$ is invertible and X posseses the following general form :

$$X = CW_1 (W_2 DACW_1)^{-1} W_2D. \qquad \dots (2.3)$$

PROOF : (*a*) This statement can be proved as in [8, Theorem 2.1.1 and Lemma 2.5.2].

(*b*) If X has the form (2.2), then it is easy to verify $X \in A\ \{1, 2, 3\}$. We need to prove that the form (2.2) holds for all $\{1, 2, 3\}$ inverses of A. Indeed, if $X \in A\ \{1, 2, 3\}$, then $X = Q_r^{-1} P_l^{-1}$, and from eq. (3) it follows that $(PP_l^{-1})^* = PP_l^{-1}$. Thus $P^* PP_l^{-1} = P^*$. The operator $P^* P$ is invertible, so that $P_l^{-1} = (P^* P)^{-1} P^*$. The right inverse of Q retains the general form

$Q_r^{-1} = W_1 (QW_1)^{-1}$ given in (2.1). Consequently,

$$X = W_1 (QW_1)^{-1} (P^*P)^{-1} P^*.$$

(*c*) This part of the proof can be proved in the same way as (*b*).

(*d*) Follows from (*b*) and (*c*) (also, this fact is proved in [2]).

(*e*) If X possesses the form (2.3), it is not difficult to verify $X \in A\ \{2\}$. On the other hand, using the method from [8, Theorem 3.4.1], it is easy to verify that $X \in A\ \{2\}$ if and only if there exist operators C and D, such that DAC is g-invertible and

$$X = C(DAC)^{(1,2)} D,\ C \in B(\mathcal{H}_4, \mathcal{H}_1), D \in B(\mathcal{H}_2, \mathcal{H}_3).$$

According to part (*a*), $X \in A\ \{2\}$ if and only if there exist operators $W_1 \in B(\mathcal{H}_5, \mathcal{H}_4)$ and $W_2 \in B(\mathcal{H}_3, \mathcal{H}_5)$, such that W_2DACW_1 is invertible, and X possesses the form (2.3). □

Lemma 2.2 — Let χ be a Banach space. If $A \in B(\chi)$, $1 \geq k$ = asc (A) = des $(A) < \infty$ and $A^l = P_{A^l} Q_{A^l}$ is the full-rank decomposition of A^l, then

$$A^D = P_{A^l} (Q_{A^l} A P_{A^l})^{-1} Q_{A^l}.$$

PROOF : If asc (A) = des (A) = $k < \infty$, then it is well-known that $\mathcal{N}(A^l) = \mathcal{N}(A^k)$ and $\mathcal{R}(A^l) = \mathcal{R}(A^k)$ for all $l \geq k$,

$$\chi = \chi_1 \oplus \chi_2, \qquad \text{... (2.4)}$$

where $\chi_1 = \mathcal{N}(A^l)$ and $\chi_2 = \mathcal{R}(A^l)$, $A(\chi_i) \subset \chi_i$ for i = 1, 2, $A_1 = A|_{\chi_1}$ is nilpotent and $A_2 = A|_{\chi_2}$ is invertible (A is not nilpotent)[3, 4]. We can write

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix}$$

with respect to the decomposition (2.4) (see [3]). Since $\mathcal{N}(A^l)$ and $\mathcal{R}(A^l)$ are complementary and closed subspaces of χ, it follows that A^l is g-invertible, so there exists a full-rank decomposition $A^l = P_{A^l} Q_{A^l}$, where $P_{A^l} \in B(Z, \chi)$ is left invertible and $Q_{A^l} \in B(\chi, Z)$ is right invertible, for some Banach space Z. By the isomorphism theorem[3], we can take that $Z = \chi_2$. We conclude that P_{A^l} and Q_{A^l} have the following representations with respect to (2.4) :

$$P_{A^l} = \begin{bmatrix} M \\ \tilde{P} \end{bmatrix} \text{ and } Q_{A^l} = [N\ \tilde{Q}],$$

where $\tilde{P}, \tilde{Q} \in B(\chi_2)$, $M \in B(\chi_2, \chi_1)$, $N \in B(\chi_1, \chi_2)$. Now, P_{A^l} is left invertible and Q_{A^l} is right invertible, so P_{A^l} and Q_{A^l} are g-invertible operations, $\mathcal{N}(P_{A^l}) = \{0\}$ and $\mathcal{R}(Q_{A^l}) = \chi_2$. It follows that $\mathcal{R}(P_{A^l}) = \mathcal{R}(A^l) = \chi_2$ and $\mathcal{N}(Q_{A^l}) = \mathcal{N}(A^l) = \chi_1$, so $M = 0$, $N = 0$ and

$$P_{A^l} = \begin{bmatrix} 0 \\ \tilde{P} \end{bmatrix} \text{ and } Q_{A^l} = [0 \ \tilde{Q}].$$

It is easy to verify that $\tilde{P}$ is left invertible and $\tilde{Q}$ is the right invertible in $B(\chi_2)$. But

$$\begin{bmatrix} 0 & 0 \\ 0 & A_2^l \end{bmatrix} = A^l = P_{A^l} Q_{A^l} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}\tilde{Q} \end{bmatrix},$$

so $A_2^l = \tilde{P}\tilde{Q}$. Since A_2^l is invertible, it follows that $\tilde{P}$ and $\tilde{Q}$ are invertible in $B(\chi_2)$.

Now, $Q_{A^l} A P_{A^l} = \tilde{Q} A_2 \tilde{P}$ is invertible in $B(\chi_2)$, so

$$A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}(\tilde{Q} A_2 \tilde{P})^{-1} \tilde{Q} \end{bmatrix} = P_{A^l} (Q_{A^l} A P_{A^l})^{-1} Q. \qquad \square$$

Remark 2.1 : The result of part (*e*) of Lemma 2.1 is an extension of the analogous result, introduced in [10, Theorem 2.1], stated for the set of complex matrices. Also, the result of Lemma 2.2 is an extension of an analogous result [10, Theorem 2.2], which is derived for complex matrices.

Our main aim is an application of considered general representations in a generalization of the Groetsch representation theorem.

We begin with the result which enables us to get various reflexive generalized inverses of the considered operator, changing initial operators W_1 and W_2.

Theorem 2.1 — *Let* $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ *has closed range,* $A = PQ$ *be the fullrank decomposition of* A *and* $W_1 \in B(\mathcal{H}_2, \mathcal{H}_3)$. *Suppose that* QW_1 *is right invertible,* $W_2 P$ *is left invertible,* $W = W_2 A W_1$ *and* $\tilde{W} = W^* W|_{\mathcal{R}(W^*)}$. *If* Ω *is an open set with* $\sigma(\tilde{W}) \subset \Omega \subset (), \infty)$, *and* $\{S_\beta(x)\}_\beta$ *is a family of continuous real valued functions on* Ω, *with* $\lim_\beta S_\beta(x) = \frac{1}{x}$ *uniformly on* $\sigma(\tilde{W})$, *then :*

$$X = \lim_\beta W_1 [S_\beta(\tilde{W})] W^* W_2 \in A\{1, 2\},$$

where the convergence is in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$.

Furthermore,

$$\| W_1 S_\beta(\tilde{W}) W^* W_2 - X \| \leq \| W_1 \| \sup_{x \in \sigma(\tilde{W})} | x S_\beta(x) - 1 | \cdot \| W^\dagger \| \| W_2 \| .$$

PROOF : Since $W = (W_2 P)(QW_1)$, QW_1 is onto, $W_2 P$ is one-to-one and $\mathcal{R}(W_2 P)$ is closed, it follows that $\mathcal{R}(W) = \mathcal{R}(W_2 P)$, so we may apply Theorem 1.1 for W instead of T. We conclude

$$X = \lim_\beta W_1 [S_\beta(\tilde{W})] W^* W_2 = W_1 (W_2 A W_1)^\dagger W_2 = W_1 ((W_2 P)(QW_1))^\dagger W_2.$$

Operators $W_2 P$ and QW_1 for the full-rank decomposition for W, and applying the part (*d*) of Lemma 2.1 we immediately obtain $((W_2 P)(QW_1))^\dagger = (QW_1)^\dagger (W_2 P)^\dagger$. Since $(QW_1)^\dagger$ is the right inverse of QW_1 and $(W_2P)^\dagger$ is the left inverse for W_2P, we easily conclude that

$$X = W_1(QW_1)^\dagger (W_2P)^\dagger W_2 \in A\ \{1, 2\}. \qquad \square$$

Using Lemma 2.1, similar results can be stated for $\{i, j, k\}$ generalized inverse. For example, if $W_1 = Q^*$ then $X \in A\ \{1, 2, 3\}$. Also if $W_2 = P^*$ then $X \in A\ \{1, 2, 4\}$. To avoid repetition we omit the proof.

Applying Lemma 2.1, Lemma 2.2 and the method from Theorem 2.1, we get the following representations of $\{2\}$, $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$ inverses, the Moore-Penrose inverse and the Drazin inverse.

Corollary 2.1 — Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range and $A = PQ$ be the full-rank decomposition of A according to (1.1). Let $\{S_\beta(x)\}^\beta$ be a family of continuous real valued functions on $(0, +\infty)$, with $\lim_\beta S_\beta(x) = \frac{1}{x}$ uniformly on all compact subsets of $(0, +\infty)$. Then :

(*a*) $X \in A\ \{1, 2\}$ if and only if there exist operators $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, such that QW_1 and W_2P are invertible, and

$$X = \lim_\beta W_1 [S_\beta(\tilde{W})]\, W^* W_2 = W_1\, \tilde{W}^{-1}\, W^* W_2,\ W = W_2AW_1.$$

(*b*) $X \in A\ \{1, 2, 3\}$ if and only if there exists $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ such that W_2P is invertible and

$$X = \lim_\beta\ [S_\beta(W_2AQ^*)]\, (W_2AQ^*)^*\, W_2 = Q^*\, (\widetilde{W_2AQ^*})^{-1}\, (W_2AQ^*)^* W_2.$$

(*c*) $X \in A\ \{1, 2, 4\}$ if and only if there exists $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$ such that QW_1 is left invertible and

$$X = \lim_\beta W_1\, [S_\beta(\widetilde{P^*AW_1})]\, (P^*AW_1)^* P^* = W_1(\widetilde{P^*AW_1})^{-1}\, (P^*AW_1)^*\, P^*.$$

(*d*) $$A^\dagger = \lim_\beta Q^*\, [S_\beta(\widetilde{P^*AQ^*})]\, (P^*AQ^*)^*\, P^* = Q^*\, (\widetilde{P^*AQ^*})^{-1}\, (P^*AQ^*)^*\, P^*.$$

(*e*) $X \in A\ \{2\}$ if and only if there exist operators

$$C \in B(\mathcal{H}_4, \mathcal{H}_1), D \in B(\mathcal{H}_2, \mathcal{H}_3), W_1 \in B(\mathcal{H}_5, \mathcal{H}_4), W_2 \in (\mathcal{H}_3, \mathcal{H}_5),$$

such that DAC is g-invertible, W_2DACW_1 is invertible and

$$X = \lim_\beta CW_1\, [S_\beta(\widetilde{W_2DACW_1})]\, (W_2DACW_1)^*\, W_2D.$$

(*f*) If $l \geq k = \text{ind}(A)$ and $Q_{A^l}AP_{A^l}$ is nonsingular, then

$$A^D = \lim_\beta P_{A^l} [S_\beta (\widetilde{Q_{A^l} AP_{A^l})] (Q_{A^l} AP_{A^l})^* Q;$$

The convergence is in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$.

Our aim is to use various initial conditions for W_1 and W_2, so we need the next result.

Theorem 2.2 — *Let* $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ *has closed range, let* Ω *be an open set with* $\sigma(T^* T|_{\mathcal{R}(T^*)}) \cup \sigma(TT^*|_{\mathcal{R}(T)}) \subset \Omega \subset (0, \infty)$, *and let* $\{S_\beta(x)\}^\beta$ *be a family of continuous real valued functions on* Ω, *with* $\lim_\beta S_\beta(x) = \frac{1}{x}$ *uniformly on* $\sigma(T^*T|_{\mathcal{R}(T^*)}) \cup \sigma(TT^*|_{\mathcal{R}(T)})$. *Then*

$$\lim_\beta T^* [S_\beta (TT^*|_{\mathcal{R}(T)})] = \lim_\beta [S_\beta (T^*T|_{\mathcal{R}(T^*)})] T^* = T^\dagger.$$

PROOF : Using the Weierstrass Approximation Theorem, we get that the operator $S_\beta(T^*T|_{\mathcal{R}(T^*)})$ is selfadjoint on $\mathcal{R}(T^*)$ and $S_\beta(T^*T|_{\mathcal{R}(T^*)})$ is selfadjoint on $\mathcal{R}(T)$. By Theorem 1.1 we get

$$\lim_\beta T^* [S_\beta (TT^*|_{\mathcal{R}(T)})] = \lim_\beta [S_\beta (T^*T|_{\mathcal{R}(T^*)})] T)^* = ((T^*)^\dagger)^* = T^\dagger$$

$$= \lim_\beta [S_\beta (T^* T|_{\mathcal{R}(T^*)})] T^*.$$ □

In the following theorem we obtain a few additional initial conditions for the operators W_1 and W_2, which produce various subsets of $\{i, j, k\}$ generalized inverses.

Theorem 2.3 — Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $A = PQ$ be a full-rank decomposition of A, $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ and $W = W_2 A W_1 \in B(\mathcal{H}_3)$.

(*a*) If W_2 is unitary, QW_1 is right invertible and S_β is a family possessing the properties from Theorem 1.1 with $T = AW_1$, then

$$\lim_\beta W_1 [S_\beta (\tilde{W})] W^* W_2 = W_1 (AW_1)^\dagger \in A \{1, 2, 3\}.$$

(*b*) If W_1 is unitary, $W_2 P$ is left invertible and S_β is a family which satisfies conditions of Theorem 1.1 for the operator $T = A^* W_2^*$, then

$$\lim_\beta W_1 W^* [S_\beta (\tilde{W})] W_2 = (W_2 A)^\dagger W_2 \in A \{1, 2, 4\}.$$

(*c*) If both W_1 and W_2 are unitary and S_β has the properties from (*a*) and (*b*), then

$$A^\dagger = \lim_\beta W_1 [S_\beta (\tilde{W})] W^* W_2 = W_1 (AW_1)^\dagger$$

$$= \lim_{\beta} W_1 W^* [S_\beta(\tilde{W})] W_2 = (W_2 A)^\dagger W_2.$$

(*d*) If (*a*) is valid and $W_1 = Q^*$, then

$$\lim_{\beta} W_1 [S_\beta(\tilde{W})] W^* W_2 = Q^* (AQ^*)^\dagger = A^\dagger.$$

(*e*) If (*b*) is valid and $W_2 = P^*$, then

$$\lim_{\beta} W_1 W^* [S_\beta(\tilde{W})] W_2 = (P^* A)^\dagger P^* = A^\dagger.$$

PROOF : (*a*) The operator W_2 is unitary, which implies

$$W^* W = (AW_1)^* AW_1,\ W^* W_2 = (AW_1)^*.$$

Since $W^* = (AW_1)^* W_2^*$ and W_2 is invertible, it follows that $\mathcal{R}(W^*) = \mathcal{R}((AW_1)^*)$. Using Theorem 1.1 we obtain

$$X = \lim_{\beta} W_1 [S_\beta(\tilde{W})] W^* W_2 = W_1 (AW_1)^\dagger \in A\ \{2, 3\}.$$

We need to prove $W_1 (AW_1)^\dagger \in A\ \{1\}$. Note that $X = W_1\ [P(QW_1)]^\dagger$. Now, P is left invertible, and QW_1 is right invertible, so $P(QW_1)$ is given as the full-rank factorization. Using the result from Lemma 2.1 (*d*) or [2], we get $X = W_1 (QW_1)^\dagger P^\dagger$. Now, eq. (1) can be easily verified.

(*b*) We use (*a*) with : A^* instead of A, W_2^* instead of W_1 and W_1^* instead of W_2. Note that W_1^* is unitary and $(W_2 P)^* = P^* W_2^*$ is right invertible. In this case we have

$$W_1 W^* = A^* W_2^*,\ WW^* = W_2 A(W_2 A)^*,\ \mathcal{R}(W_2 A) = \mathcal{R}(W)$$

which implies $\tilde{W}^* = (\tilde{W}_2 A)^*$. Using the Weierstrass Approximation Theorem, we get that $S_\beta(W_2 A(W_2 A)^* |_{\mathcal{R}(W_2 A)})$ is selfadjoint, so

$$\lim_{\beta} (A^* W_2^*) [S_\beta(W_2 A(W_2 A)^* |_{\mathcal{R}(W_2 A)})]\ W_2$$

$$= \lim_{\beta} \left\{ W_2^* [S_\beta((A^* W_2^*)^* A^* W_2^* |_{\mathcal{R}[(A^* W_2^*)^*]})] (A^* W_2^*)^* \right\}^* = (W_2^* (A^* W_2^*)^\dagger)^*.$$

By (*a*) we know that $W_2^* (A^* W_2^*)^\dagger \in A^*\ \{1, 2, 3\}$, so

$$(W_2^* (A^* W_2^*)^\dagger)^* = (W_2 A)^\dagger W_2 \in A\ \{1, 2, 4\}.$$

(*c*) It is enough to prove that the limits from (*a*) and (*b*) are equal. If W_2 is unitary, from the proof of (*a*) we get $\tilde{W}=A\tilde{W}_1$. If W_1 is unitary, from the proof of (*b*) we get $\tilde{W}^* = (\tilde{W}_2 A)^*$. Now, by Theorem 2.4, and using the parts (*a*) and (*b*) of this proof, we get :

$$W_1(AW_1)^{\dagger} = \lim_{\beta} W_1 [S_{\beta}((AW_1)^* AW_1 |_{\mathcal{R}[(AW_1)^*]}] (AW_1)^* W_2^* W_2$$

$$= \lim_{\beta} W_1 [S_{\beta}(W^*W |_{\mathcal{R}(W^*)})] W^*W_2 = W_1 W^{\dagger} W_2$$

$$= \lim_{\beta} W_1 W^* [S_{\beta}(WW^* |_{\mathcal{R}(W)})] W_2 = \lim_{\beta} A^* W_2^* [S_{\beta}(WW^* |_{\mathcal{R}(W)})] W_2$$

$$= \lim_{\beta} A^* W_2^* S_{\beta}(W_2A(W_2A)^* |_{\mathcal{R}(W_2A)}) W_2 = (W_2A)^{\dagger} W_2. \qquad \square$$

Finally, as corollaries, we introduce a few iterative methods for computing relflexive g-inverses.

Corollary 2.2 — Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $A = PQ$ is the full rank decomposition of A and let $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ be two operators, such that QW_1 is right invertible and $W_2 P$ is left invertible. Let $W = W_2 A W_1$ and $\tilde{W} = W^* W |_{\mathcal{R}(W^*)}$. Then the following representations of the reflexive g-inverses are convergent in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$:

(*a*) $$A^{(1,2)} = W_1 \left[\int_0^{\infty} e^{-W^*W_u} W^* \, du \right] W_2;$$

(*b*) $$A^{(1,2)} = \alpha W_1 \sum_{k=0}^{\infty} (I - \alpha W^*W)^k W^* W_2, \text{ where } 0 < \alpha < 2 \| W \|^{-2};$$

(*c*) $$A^{(1,2)} = W_1 \lim_{t \to 0_+} (tI + W^*W)^{-1} W^* W_2;$$

(*d*) $$A^{(1,2)} = W_1 \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\prod_{j=0}^{k-1} \left(1 - \frac{1}{j+1} W^*W \right) \right) W^* W_2;$$

(*e*) $$A^{(1,2)} = W_1 \lim_{t \to 0_+} \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+tk)} [I - W^*W]^k W^* W_2;$$

(*f*) $$A^{(1,2)} = W_1 \left(W^* + \lim_{t \to 0_+} \sum_{k=1}^{\infty} e^{-tk \log k} [1 - W^*W]^k W^* \right) W_2; \text{ and}$$

(*g*) $A^{(1,2)} = W_1 \lim_{t \to 0_+} \sum_{k=0}^{\infty} \frac{\Gamma(1 = 1 - t)k)}{\Gamma(1 + k)} [I - W^*W] W^*W_2.$

Also, as a corollary, we get the next generalization of the main result form :[9]

Corollary 2.3 ([9, Lemma 2.1, Theorem 2.1]) — Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $A = PQ$ is the full-rank decomposition of A and let $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ be two operators, such that QW_1 is invertible and W_2P is invertible. Let $W = W_2AW_1$. Then the class of $\{1, 2\}$ inverse of A can be generated by changing the values of the operators W_1, W_2 in the following two iterative processes :

$$Y_0 = Y_0' = \alpha (W_2AW_1)^* \quad 0 < \alpha \leq 2 \| W \|^{-2},$$

$$\begin{cases} T_k = I_X - Y_k W, \\ Y_{k+1} = (I_X + T_k + \ldots + T_k^{q-1}) Y_k, \\ X_{k+1} = W_1 Y_{k+1} W_2 \end{cases} , \qquad \begin{cases} T_k' = I_X - WY_k', \\ Y'_{k+1} = Y_k' (I_Y + T_k' + \ldots + T_k'^{q-1}), \\ X_{k+1}' = W_1 Y_{k+1}' W_2 \quad k = 0, 1, \ldots \end{cases}$$

Moreover, the following statements are valid :

(*a*) If W_2 is unitary, then $X_k \to X = W_1(AW_1)^\dagger \in A\ \{1, 2, 3\}$ as $k \to \infty$.

(*b*) If W_1 is unitary, then $X_k' \to X = (W_2A)^\dagger W_2 \in A\ \{1, 2, 4\}$ as $k \to \infty$.

(*c*) If (*a*) and (*b*) are valid, then $X_k \to A^\dagger$.

(*d*) If (*a*) is valid and $W_1 = Q^*$, then $X_k \to X = A^\dagger$.

(*e*) If (*b*) is valid and $W_2 = P^*$, then $X_k' \to X = A^\dagger$.

Remark 2.1 : In [6] it is also introduced a modification of the hyper-power method, which generates the class of all $\{1, 2\}$-inverses for operators on Banach spaces. Using the method from [6] for Hilbert spaces operators, it is not clear how to choose the initial values to get $\{1, 2, 3\}$, $\{1, 2, 4\}$-inverses. Also, our method is applicable for various classes of $\{S_\beta\}$ families.

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