Indian J. pure appl. Math., **31**(3) : 309-317, March 2000

CERTAIN PSEUDO-DIFFERENTIAL OPERATOR ASSOCIATED WITH THE BESSEL OPERATOR

R. S. PATHAK AND S. PATHAK

Department of Mathematics, Banaras Hindu University, Varanasi 221 005

(*Received* 13 *May* 1999; *Accepted* 22 *July* 1999)

Using inverse Hankel transform a symbol is defined, and the pseudo-differential operator (p.d.o.) $\mathcal{G}(x, D)$ associated with the Bessel operator $\frac{d^2}{dx^2} + \frac{(1-4\mu^2)}{4x^2}$ in terms of this symbol is defined. It is shown that the p.d.o. is bounded in a certain Sobolev type space associated with the Hankel transform.

Key Words : Pseudo-differential Operator; Bessel Operator; Hankel Transform; Schwartz's Theory of Fourier Transformation; Soboloev type Spaces

1. INTRODUCTION

The theory of the Hankel transformation

$$(h_\mu \phi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\, \phi(y)dy \qquad \dots (1.1)$$

has been extended by Zemanian[16] to distributions belonging to H'_μ, the dual of the test function space H_μ, consisting of all complex valued infinitely differentiable function ϕ on $I = (0, \infty)$ and satisfying

$$\gamma^\mu_{m,k}(\phi) = \sup_{x \in I} \left| x^m \left(x^{-1}\frac{d}{dx}\right)^k \left(x^{-\mu-\frac{1}{2}} \phi(x)\right)\right| < \infty, \ \forall\, m, k \in \mathbb{N}_0. \qquad \dots (1.2)$$

Zaidman[15] studied a class of pseudo-differential operators (p.d.o's) using Schwartz's theory of Fourier transformation. Pseudo-differential operators associated to a numerical valued symbol $a(x, y)$ were discussed by Pathak and Pandey[7, 9] and Pathak and Upadhyay[11]. One formula for such an operator appears as follows :

$$(h_{\mu,a} u)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\, a(x, y)\, U_\mu(y)dy, \qquad \dots (1.3)$$

where

$$U_\mu(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\, u(x)dx,\ \mu \geq -\frac{1}{2}. \quad \text{... (1.4)}$$

The symbol $a(x, y)$ is defined to be the complex valued infinitely differentiable function on $I \times I$ which satisfies

$$(1+x)^q \left| \left(x^{-1}\frac{d}{dx}\right)^\nu \left(y^{-1}\frac{d}{dy}\right)^\alpha a(x,y) \right| \leq D_{\alpha,\nu,m,q}(1+y)^{m-\alpha} \quad \text{... (1.5)}$$

for all $q, \nu, \alpha \in I\!N_0$, where m is a fixed real number. The class of all such symbols is denoted by H^m, if $a(x, y)$ satisfies (1.5) with $q = 0$, then the symbol class will be denoted by H_0^m.

We shall use the notation and terminology of [2], [4], [6], [13] and [16]. The differential operator S_μ is defined by

$$S_\mu = S_{\mu,x} = \frac{d^2}{dx^2} + \frac{(1-4\mu^2)}{4x^2}. \quad \text{... (1.6)}$$

From [16, p. 139] we know that for any $\phi \in H_\mu$,

$$h_\mu(S_\mu \phi) = -y^2 h_\mu \phi, \quad \text{... (1.7)}$$

$$\left(x^{-1}\frac{d}{dx}\right)^k (\theta \cdot \phi) = \sum_{\nu=0}^{k} \binom{K}{\nu} \left(x^{-1}\frac{d}{dx}\right)^\nu \cdot \theta \left(x^{-1}\frac{d}{dx}\right)^{k-\nu} \phi, \quad \text{... (1.8)}$$

and from [3, p. 948] we have

$$S_{\mu,x}^r \phi(x) = \sum_{j=0}^{r} b_j x^{2j+\mu+1/2} \left(x^{-1}\frac{d}{dx}\right)^{r+j} \left(x^{-\mu-1/2}\phi(x)\right) \quad \text{... (1.9)}$$

where b_j are constants depending only on μ.

Hankel convolution on certain ultra-differentiable function spaces and ultra-distribution spaces has been studied by Pathak and Shrestha[10]. The theory of Hankel transform and Hankel convolution has been used to develop a theory of the pseudo-differential operator associated with the Bessel operator[8, 11,12].

We shall also make use of a lemma due to Haimo[1] for the Hankel convolution transform which we recall here.

Let $\Delta(x, y, z)$ be the area of a triangle with sides x, y, z if such a triangle exists.

For fixed $\mu \geq -1/2$, set

$$D(x, y, z) = 2^{3\mu - 1/2} (\pi)^{-1/2} [\Gamma(\mu + 1)]^2 \left[\Gamma\left(\mu + \frac{1}{2} \right) \right]^{-1} (xyz)^{-2\mu} [\Delta(x, y, z)]^{2\mu - 1} \quad \text{... (1.10)}$$

if Δ exists and zero otherwise. We note that $D(x, y, z) \geq 0$ and that $D(x, y, z)$ is symmetric in x, y, z and from [14, p. 411],

$$j(xt)\, j(yt) = \int_0^\infty j(zt)\, D(x, y, z)\, d\sigma(z), \quad \text{... (1.11)}$$

where

$$d\sigma(x) = (2^\mu\, \Gamma(\mu + 1))^{-1}\, x^{2\mu + 1}\, dx \quad \text{... (1.12)}$$

and

$$j(x) = 2^\mu\, \Gamma(\mu + 1)\, x^{-\mu}\, J_\mu(x). \quad \text{... (1.13)}$$

From [5, p. 40] we know that

$$J_\mu(x\xi)\, J_\mu(x\lambda) = \frac{1}{2^\mu\, \Gamma(\mu + 1)} \int_0^\infty (x\lambda\xi)^\mu\, z^{-\mu}\, J_\mu(zx)\, D(\xi, \lambda, z)\, d\sigma(z). \quad \text{... (1.14)}$$

Next we define the space $L^p_\mu(I)$, $1 \leq p < \infty$, as the space of those real valued measurable functions on I for which

$$\| f \|_p = \left[\int_0^\infty | f(x) |^p\, d\sigma(x) \right]^{1/p} < \infty. \quad \text{... (1.15)}$$

We denote by $L^\infty_\mu(I)$ the space of all real valued measurable functions on I for which

$$\| f \|_\infty = \underset{0 < x < \infty}{\text{ess sup}} | f(x) | < \infty. \quad \text{... (1.16)}$$

Let $f \in L^l_\mu(I)$; then its associated function $f(x, y)$ is defined by

$$f(x, y) = \int_0^\infty f(z)\, D(x, y, z)\, d\sigma(z), \; 0 < x, y < \infty. \quad \text{... (1.17)}$$

Lemma 1.1 (Haimo) — Let f and g be functions of $L^1_\mu(I)$ and let

$$f \# g(x) = \int_0^\infty f(x, y)\, g(y)\, d\sigma(y), \; 0 < x < \infty. \quad \text{... (1.18)}$$

Then the integral defining $f \# g(x)$ converges for almost all x, $0 < x < \infty$, and

$$\| f \# g \|_{L^1} \le \| f \|_{L^1} \| g \|_{L^1}. \qquad \dots (1.19)$$

In this paper the p.d.o. $\mathcal{G}(x, D)$ is defined by means of a symbol which is assumed to be the Hankel transform of a function satisfying certain boundedness condition. It is shown that the p.d.o. satisfies certain L^1-norm inequality. Some special cases are discussed.

2. PSEUDO-DIFFERENTIAL OPERATOR $\mathcal{G}(x, D)$

We assume that the symbol $a(x, \xi)$ is defined as the Hankel transform :

$$a(x, \xi) = x^{-\mu - 1/2} \int_0^\infty (x\lambda)^{1/2} J_\mu(x\lambda) V(\lambda, \xi) \, d\lambda, \qquad \dots (2.1)$$

where $V(\lambda, \xi)$ is a complex valued measurable function on $I \times I$, such that $V(\lambda, \xi)$ is λ-measurable $\forall\ \xi \in I$, and

$$| V(\lambda, \xi) | \le k(\lambda) \ \ \forall\ \xi \in I, \ \forall\ \lambda \in I, \qquad \dots (2.2)$$

where $k(\lambda) \in L^1(I)$, $\mu \ge -1/2$. Since $\left| x^{1/2} J_\mu(x) \right| \le A_\mu$ for $\mu \ge -\frac{1}{2}$, the integral (2.1) exists under the assumption (2.2).

Let G denote the set of all $a(x, \xi)$ on $I \times I$ such that (2.1) and (2.2) hold. Now, for $a(x, \xi) \in G$ define the p.d.o.

$$\mathcal{G}(x, D)\, u(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\, a(x, \xi)\, U_\mu(\xi)\, d\xi \qquad \dots (2.3)$$

where

$$U_\mu(\xi) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\, u(x)\, dx, \ \mu \ge -\frac{1}{2}. \qquad \dots (2.4)$$

We justify the above definition of $\mathcal{G}(x, D)$ by proving the existence of the integral (2.3). Since

$$a(x, \xi) = x^{-\mu - 1/2} \int_0^\infty (x\lambda)^{1/2} J_\mu(x\lambda)\, V(\lambda\xi)\, d\lambda,$$

therefore for $\left| x^{1/2} J_\mu(x) \right| \le A_\mu$, we have

$$|\mathcal{G}(x, D)u(x)| = \left| \int_0^\infty (x\xi)^{\frac{1}{2}} J_\mu(x\xi) (x^{-\mu-1/2} \int_0^\infty (x\lambda)^{1/2} J_\mu(x\lambda) V(\lambda, \xi)\, d\lambda)\, U_\mu(\xi)\, d\xi \right|$$

$$= \left| x^{-\mu-\frac{1}{2}} \right| \left| \int_0^\infty (x\lambda)^{\frac{1}{2}} J_\mu(x\lambda) d\lambda \int_0^\infty (x\xi)^{\frac{1}{2}} J_\mu(x\xi) V(\lambda, \xi) U_\mu(\xi)\, d\xi \right|$$

$$\leq \int_0^\infty \int_0^\infty A_\mu^2 x^{-\mu-\frac{1}{2}} |V(\lambda, \xi)| |U_\mu(\xi)|\, d\lambda\, d\xi$$

$$\leq x^{-\mu-\frac{1}{2}} A_\mu^2 \int_0^\infty |k(\lambda)|\, d\lambda \int_0^\infty |U_\mu(\xi)|\, d\xi < \infty,$$

because $k(\lambda) \in L^1(I)$ and $U_\mu(\xi) \in H_\mu(I)$.

Now we prove a boundedness result for $\mathcal{G}(x, D)$ for which we need the following Sobolev type space.

Definition 2.1 — *(Sobolev Type Space) — The space* $\mathcal{G}^s_{\mu, p}(I)$, $s \in \mathbb{R}$ $\mu \in \mathbb{R}$, *is defined to be the set of all those elements* $u \in H'_\mu(I)$ *which satisfy*

$$\| u \|_{\mathcal{G}^s_{\mu, p}} = \left\| \eta^{s-\mu-1/2} h_\mu u \right\|_p, \; 1 \leq p < \infty. \quad \ldots (2.5)$$

Theorem 2.2 — *Let* $\mu \geq -1/2$. *Then*

$$\| \mathcal{G}(x, D)\, u \|^0_{\mathcal{G}_{\mu,1}} \leq \| K \|_{L^1} \| u \|_{\mathcal{G}^0_{\mu, 1}}, \; u \in H_\mu(I). \quad \ldots (2.6)$$

PROOF : We have

$$\mathcal{G}(x, D)\, u(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\, a(x, \xi)\, (h_\mu u)(\xi)\, d\xi,$$

where

$$a(x, \xi) = x^{-\mu-1/2} \int_0^\infty (x\lambda)^{1/2} J_\mu(x\lambda)\, V(\lambda, \xi)\, d\lambda.$$

Therefore, by changing the order of integrations using Fubini's theorem, we have

$$\mathcal{G}(x, D)\, u(x)$$

$$= \int_0^\infty (x\xi)^{\frac{1}{2}} J_\mu(x\xi) \left(x^{-\mu-\frac{1}{2}} \int_0^\infty (x\lambda)^{\frac{1}{2}} J_\mu(x\lambda)\, V(\lambda, \xi)\, d\lambda \right) U_\mu(\xi)\, d\xi$$

$$= \int_0^\infty \int_0^\infty x^{-\mu-\frac{1}{2}} (x\xi)^{\frac{1}{2}} (x\lambda)^{\frac{1}{2}} V(\lambda, \xi)\, U_\mu(\xi) \frac{1}{2^\mu \Gamma(\mu+1)} \int_0^\infty (x\lambda\xi)^\mu z^{-\mu}$$

$$\times J_\mu(zx)\, D(\xi, \lambda, z)\, d\sigma(z)\, d\lambda d\xi$$

$$= \int_0^\infty \int_0^\infty x^{-\mu-\frac{1}{2}} (x\xi)^{\frac{1}{2}} (x\lambda)^{\frac{1}{2}} V(\lambda, \xi)\, U_\mu(\xi) \frac{1}{2^\mu \Gamma(\mu+1)} \int_0^\infty (x\lambda\xi)^\mu$$

$$\times z^{-\mu} J_\mu(zx)\, D(\xi, \lambda, z) \frac{1}{2\mu\, \Gamma(\mu+1)} z^{2\mu+1}\, dz\, d\lambda\, d\xi$$

$$= \frac{1}{(2^\mu \Gamma(\mu+1))^2} \int_0^\infty (zx)^{\frac{1}{2}} J_\mu(zx) \left[\int_0^\infty \int_0^\infty \xi^{\mu+\frac{1}{2}} U_\mu(\xi)\, \lambda^{\mu+\frac{1}{2}} V(\lambda, \xi)\, D(\xi, \lambda, z)\, d\lambda d\xi \right] \cdot z^{\mu+\frac{1}{2}}\, dz.$$

An application of the inverse Hankel transform yields

$$\int_0^\infty (zx)^{\frac{1}{2}} J_\mu(zx)\, \mathcal{G}(x, D)\, u(x)\, dx$$

$$= \frac{z^{\mu+\frac{1}{2}}}{(2^\mu \Gamma(\mu+1))^2} \int_0^\infty \int_0^\infty \xi^{\mu+\frac{1}{2}} U_\mu(\xi)\, \lambda^{\mu+\frac{1}{2}} V(\lambda, \xi)\, D(\xi, \lambda, z)\, d\lambda\, d\xi.$$

In other words, we have

$$h_\mu(\mathcal{G}(x, D)\, u(x))\, (z)$$

$$= z^{\mu+\frac{1}{2}} \int_0^\infty \int_0^\infty \lambda^{-\mu-\frac{1}{2}} \xi^{-\mu-\frac{1}{2}} V(\lambda, \xi)\, D(\xi, \lambda, z)\, U_\mu(\xi)\, d\sigma(\lambda)\, d\sigma(\xi).$$

Using the inequality (2.2) we obtain

$$\left| h_\mu(\mathcal{G}(x, D)\, u(x))\, (z) \right|$$

$$\le z^{\mu+\frac{1}{2}} \int_0^\infty \int_0^\infty \lambda^{-\mu-\frac{1}{2}} k(\lambda)\, D(\xi, \lambda, z)\, \xi^{-\mu-\frac{1}{2}} \,|\, U_\mu(\xi) \,|\, d\sigma(\xi)\, d\sigma(\lambda)$$

$$= z^{\mu+\frac{1}{2}} (K \# V_\mu)\, (z)$$

where

$$k(\lambda) = \lambda^{-\mu - 1/2}\, k(\lambda)$$

and

$$V_{\mu}(\xi) = \xi^{-\mu - 1/2} \mid U_{\mu}(\xi) \mid.$$

Hence

$$\int_0^\infty z^{-\mu - 1/2} \mid h_{\mu}(\mathcal{G}(x, D)\, u(x))\,(z) \mid d\sigma(z)$$

$$\leq \int_0^\infty (K \# V_{\mu})\,(z)\, d\sigma(z).$$

Now applying the definition (2.1) and (1.19) we get

$$\| \mathcal{G}(x, D)\, u(x) \|_{\mathcal{G}^{\rho}_{\mu, 1}} \leq \| K \|_{L^1} \| u \|_{\mathcal{G}^{\rho}_{\mu, 1}} .$$

3. Some Properties of Symbols

1. Assume that $x^{\mu + 1/2}\, a(x, \xi) \in L^1$ (I) for fixed $\xi \in (0, \infty)$ and $\mu \geq -1/2$. Let us define

$$V(\lambda, \xi) = \int_0^\infty (x\lambda)^{1/2} J_{\mu}(x\lambda)\, x^{\mu + 1/2}\, a(x, \xi)\, dx$$

which is the same as $A_{\xi}(\lambda)$ in the notation of Pathak and Pandey[7]. Then from [7, p. 74)] we know that for $\mu \geq -1/2$ and $r \in I\!N_0$,

$$\mid V(\lambda, \xi) \mid \leq C_{r, m, q} (1 + \xi)^m \lambda^{\mu + 1/2} (1 + \lambda^{2r})^{-1}.$$

Clearly, if $a(x, \xi) \in H^0$, then

$$\mid V(\lambda, \xi) \leq k(\lambda) \quad \forall\ \xi \in I, \ \forall\ \lambda \in I, \qquad \text{... (3.1)}$$

where

$$k(\lambda) = C_{r, q}\, \lambda^{\mu + 1/2} (1 + \lambda^{2r})^{-1} \in L^1 (I). \qquad \text{... (3.2)}$$

Conversely, let $k(\lambda) \in L^1 (I)$ and $\mid V(\lambda, \xi) \mid \leq k(\lambda)$, ... (3.3)

then by inverse Hankel transform the symbol $a(x, \xi)$ is defined by

$$a(x, \xi) = x^{-\mu - 1/2} \int_0^\infty (x, \lambda)^{1/2} J_\mu(x\lambda)\, V(\lambda, \xi)\, d\lambda. \qquad \text{... (3.4)}$$

Since $\left| x^{1/2} J_\mu(x) \right| \leq A_\mu$ for $\mu \geq -\frac{1}{2}$, the integral (3.4) exists under the assumption (3.3).

2. Let us now consider the special case when the symbol $a(x, \xi)$ is separable in the form

$$a(x, \xi) = a(x)\, b(\xi),$$

where $\lambda^{\mu + \frac{1}{2}} h_\mu(a(x))(\lambda) \in L^1(I)$ and $b(\cdot)$ is a bounded measurable function on I ($|b(\xi)| \leq M$ $\forall$ $\xi \in I$).

Since

$$x^{\mu + 1/2} a(x) = \int_0^\infty (x\lambda)^{1/2} J_\mu(x\lambda)\, h_\mu\left[x^{\mu + 1/2} a(x) \right](\lambda)\, d\lambda$$

and $\qquad \lambda^{\mu + 1/2} h_\mu[a(x)](\lambda) \in L^1(I),$

therefore,

$$x^{\mu + 1/2} a(x, \xi) = \int_0^\infty (x\lambda)^{1/2} J_\mu(x\lambda) \left[h_\mu x^{\mu + 1/2} a(x) \right](\lambda)\, b(\xi)\, d\lambda.$$

Thus

$$V(\lambda, \xi) = h_\mu\left(x^{\mu + 1/2} a(x, \xi) \right)(\lambda) = h_\mu\left(x^{\mu + 1/2} a(x) \right)(\lambda)\, b(\xi),$$

which is a measurable function on $I \times I$ $\forall$ $\xi \in I$, since

$$|b(\xi)| \leq M, \text{ and } |V(\lambda, \xi)| \leq M \left| h_\mu\left(x^{\mu + 1/2} a(x) \right)(\lambda) \right| \in L^1(I).$$

Thus (3.3) is everified with $k(\lambda) = M \left| h_\mu\left(x^{\mu + 1/2} a(x) \right)(\lambda) \right|$.

Therefore, by the preceding theorem,

$$\| \mathcal{G}(x, D) u(x) \|_{\mathcal{G}^\rho_{\mu, 1}} \leq \| K \|_{L^1} \| u \|_{\mathcal{G}^\rho_{\mu, 1}},$$

where

$$K(\lambda) = \lambda^{-\mu - 1/2} k(\lambda)$$

$$= \lambda^{-\mu - 1/2} M\, h_\mu\left(x^{\mu + 1/2} a(x) \right)(\lambda).$$

REFERENCES

1. D. T. Haimo *Trans. Amer. Math. Soc.*, **116** (1965), 330-75.
2. I. I. Hirschman *J. Anal. Math.* **8** (1960.61), 307-36.
3. E. L. Koh and A. H. Zemanian. *SIAM J appl. Math.* **16** (5) (1968), 945-57.
4. L. Schwartz, *Theorie' des Distributions*, Hermann, Paris, 1978.
5. P. K. Pandey, A class of pseudo-differential operators, *Ph. D. Thesis*, Banaras Hindu University, Varanasi (1995).
6. R. S. Pathak. *Integral Transforms of Generalized Functions and their Applications*, Gardon and Breach Science Publishers, 1997.
7. R. S. Pathak and P. K. Pandey *J. math. Anal. Appl.*, **196** (1995) 736-47.
8. R. S. Pathak and P. K. Pandey. Product of pseudo-differential operators associated with Bessel operators, (1997) (*Communicated*) .
9. R. S. Pathak and P. K. Pandey, *J. math. Anal. Appl.*, **215** (1997), 95-111.
10. R. S. Pathak and K. K. Shrestha. Hankel convolution on some ultra-differentiable function spaces, (1998) (*Communicated*)
11. R. S. Pathak and S. K. Upadhyay. *J. math. Anal. Appl.* **213** (1997), 133-47.
12. R. S. Pathak and S. K. Upadhyay. (1998) L_μ^p -boundedness of pseudo-differential operator associated with the Bessel operator. (Communicated)
13. I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill Book Co., New York (1972).
14. G. N. Watson. *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1958.
15. S. Zaidman. *Ann Mat. Pura Appl* **92** (1972), (4) 345-99.
16. A. H. Zemanian, *Generalized Integral Transformations*, Interscience, New York, 1968.