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SOME SEQUENCE SPACES DEFINED BY $|\overline{N}, p_n|$ SUMMABILITY AND AN ORLICZ FUNCTION

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The object of this paper is to introduce some new sequence spaces which arise from the notions of $|\overline{N}, p_n|$ summability and an Orlicz function M. Some topological results and certain inclusion relations between these spaces have been discussed.

Key Words : Sequence Space; Summability; Orlicz Function; Topological Linear Space; Paranorm.

1. INTRODUCTION AND PRELIMINARIES

Given an infinite series $\sum_{n=0}^{\infty} a_n$, which we will denote by "a", let

$$s_n = a_0 + a_1 + \ldots + a_n \qquad \ldots (1.1)$$

and denote the sequence (s_n) by "s".

We will suppose throughout that a, s are related by (1.1). (Where no limits are stated sums throughout are to be taken from 1 to ∞).

Denote by $(p_n)_{n \geq 0}$ a sequence of positive real numbers, and write $P_n = \sum_{k=0}^{n} p_k$.

It is well known that the series a (or the sequence s) is said to be summable $|\overline{N}, p_n|$ to the sum l (finite)

if $$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \to l \text{ as } n \to \infty,$$

and is said to be absolutely summable $(\overline{N}, p_n)$, or summable $|\overline{N}, p_n|$, if also the sequence $(t_n) \in BV$, that is $\sum_n |t_n - t_{n-1}| < \infty$.

Let $|\overline{N}_p|$ and $\overline{N}_p$, denote, respectively, the set of all sequences which are summable $|\overline{N}, p_n|$ and $(\overline{N}, p_n)$. If $p_n = 1$ for all n, then $|\overline{N}_p|$ and $\overline{N}_p$, respectively, become $|C_1|$ (the set of all sequences which are absolutely Cesàro summable of order 1) and C_1 (the set of all sequences which are Cesàro summable of order 1).

We write, for $n \geq 1$,

$$\phi_n(a) = t_n - t_{n-1}, \text{ where } t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v.$$

Applying Abel's transformation to t_n and t_{n-1}, we obtain

$$t_n - t_{n-1} = \frac{1}{P_n} \left(\sum_{v=0}^{n-1} P_v \Delta S_v + P_n s_n \right)$$

$$- \left(\sum_{v=0}^{n-2} P_v \Delta s_v + P_{n-1} s_{n-1} \right); \Delta s_v = s_v - s_{v+1} = -a_{v+1}$$

$$= (s_n - s_{n-1}) + \frac{1}{P_n} \sum_{v=0}^{n-1} P_v \Delta s_v - \frac{1}{P_{n-1}} \sum_{v=0}^{n-2} P_v \Delta s_v$$

$$= a_n - \frac{1}{P_n} \sum_{v=0}^{n-1} P_v a_{v+1} + \frac{1}{P_{n-1}} \sum_{v=0}^{n-1} P_v a_{v+1} - a_n$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{v=0}^{n-1} P_v a_{v+1}$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n} P_{k-1} a_k$$

Thus we have

$$\phi_n(a) = t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n} P_{k-1} a_k \; (n \geq 1). \qquad \text{... (1.2)}$$

Note that for any series a, b and scalar λ, we have

$$\phi_n(a+b) = \phi_n(a) + \phi_n(b) \text{ and } \phi_n(\lambda a) = \lambda \phi_n(a). \qquad \text{... (1.3)}$$

Recently, Bhardwaj and Singh[1] have introduced and examined some topological properties of a new sequence space $|\overline{N}_p|(r)$ which emerges naturally as an extension of $|\overline{N}_p|$ in the same was as l, the space of absolutely convergent sequences is extended to $l(p)$ (see Bourgin[3], Landsberg[7], Maddox[10] and Simons[17]).

Definition 1.1[1] — Let $r=(r_n)$ be a bounded sequence of positive real numbers, we define

$$|\overline{N}_p|(r) = \{a=(a_n) : \sum_n |\phi_n(a)|^{r_n} < \infty\}$$

Note that, if $p_n = r_n = 1$ for all n, then $|\overline{N}_p|(r)$ reduces to $|C_1|$ and if $p_n = 1$ for all n, then $|\overline{N}_p|(r) = |C_1|(r)$ (Nanda and Mohanty[13]).

Following Lindenstrauss and Tzafriri[8], we recall that an Orlicz function M is a continuous, convex, non-decreasing function defined for $x \geq 0$ such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$.

If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called a modulus function, defined and discussed by Nakano[12], Ruckle[16], Maddox[11] and others.

Lindenstrauss and Tzafriri[8] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x=(x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space l_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(x) = x^p$; $1 \leq p < \infty$, the spaces l_M coincide with the classical sequence spaces l_p.

Recently, Parashar and Choudhary[15] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence space l_M and strongly summable sequence spaces $[C, 1, p]$, $[C, 1, p]_0$ and $[C, 1, p]_\infty$. It may be noted here that the spaces of strongly summable sequences were discussed by Maddox[9]. Nuray and Gülcü[14], Demirci[4] and others have also used an Orlicz function to construct some sequence spaces.

Quite recently, Bhardwaj and Singh[2] have introduced and studied a new concept of absolute Cesàro summability of order 1 with respect to an Orlicz function.

We now introduce the generalizations of the spaces of $|\overline{N}, p_n|$ summable sequences.

Definition 1.2 — Let M be an Orlicz function and $r=(r_n)$ be a bounded sequence of strictly positive real numbers. Define

$$|\overline{N}_p|(M) = \left\{ a=(a_n) : \sum_n M\left(\frac{|\phi_n(a)|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\},$$

$$|\overline{N}_p|(M,r)=\left\{a=(a_n):\sum_n\left[M\left(\frac{|\phi_n(a)|}{\rho}\right)\right]^{r_n}<\infty \text{ for some } \rho>0\right\}.$$

If $r_n=1$ for all n, then $|\overline{N}_p|(M,r)=|\overline{N}_p|(M)$.

$|\overline{N}_p|(M,r)$ is the generalization of several known sequence spaces, for instance the following classes arise from $|\overline{N}_p|(M,r)$ as the special cases :

(i) If $M(x) = x$, then $|\overline{N}_p|(M,r) = |\overline{N}_p|(r)$ (Bhardwaj and Singh[1]).

(ii) If $p_n=1$ for all n, then $|\overline{N}_p|(M,r) = |C_1|(M,r)$ (Bhardwaj and Singh[2]).

(iii) If $M(x) = x$ and $r_n=1$ for all n, then $|\overline{N}_p|(M,r) = |\overline{N}_p|$.

(iv) If $M(x) = x$ and $p_n=r_n=1$ for all n, $|\overline{N}_p|(M,r) = |C_1|$.

(v) If $M(x) = x$ and $p_n=1$ for all n, then $|\overline{N}_p|(M,r) = |C_1|(r)$ (Nanda and Mohanty[13]).

(vi) If $p_n=r_n=1$ for all n, then $|\overline{N}_p|(M,r) = |C_1|(M)$ (Bhardwaj and Singh[2]).

In this paper, we propose to study the linear topological structure of the sequence space $|\overline{N}_p|(M,r)$. Certain inclusion relations between $|\overline{N}_p|(M,r)$ spaces have also been discussed.

The following inequalities (see, for example, Maddox[10]) are needed throughout the paper.

Let $r=(r_k)$ be a bounded sequence of strictly positive real numbers. If H = sup r_k, then

$$|a_k+b_k|^{r_k}\le C\,(|a_k|^{r_k}+|b_k|^{r_k}), \qquad \text{... (1.4)}$$

where C = max $(1, 2^{H-1})$. Also for any complex λ,

$$|\lambda|^{r_k}\le \max(1,|\lambda|^H). \qquad \text{... (1.5)}$$

2. Linear Topological Structure of $|\overline{N}_p|(M,r)$

In this section we propose to study linear topological structure of $|\overline{N}_p|(M,r)$.

Theorem 2.1 — *For any Orlicz function M and a bounded sequence $r=(r_n)$ of strictly positive real numbers, $|\overline{N}_p|(M,r)$ is a linear space over the set of complex numbers.*

The proof is a routine verification by using 'standard' techniques and hence is omitted.

Theorem 2.2 — *For any Orlicz function M and a bounded sequence $r=(r_n)$ of strictly positive real numbers, $|\overline{N}_p|(M,r)$ is a topological linear space, totally paranormed by*

$$g(a)=\inf\left\{\rho^{r_m/H}:\left(\sum_n\left[M\left(\frac{|\phi_n(a)|}{\rho}\right)\right]^{r_n}\right)^{1/H}\le 1,\ m=1,2,\ldots\right\}$$

where H = max (1, sup r_n).

PROOF : Clearly $g(a) = g(-a)$. By using Theorem 2.1 for $\lambda = \mu = 1$, we get $g(a+b) \leq g(a) + g(b)$. Since $M(0) = 0$, we get $\inf \{\rho^{r_m/H}\} = 0$ for $a = 0$. Conversely, suppose $g(a) = 0$, then for a given $\varepsilon > 0$, there exists some ρ_ε $(0 < \rho_\varepsilon < \varepsilon)$ such that

$$\left(\sum_n \left[M\left(\frac{|\phi_n(a)|}{\varepsilon} \right) \right]^{r_n} \right)^{1/H} \leq \left(\sum_n \left[M\left(\frac{|\phi_n(a)|}{\rho_\varepsilon} \right) \right]^{r_n} \right)^{1/H} \leq 1.$$

Suppose $a_{n_m} \neq 0$ for some m. Let $\varepsilon \to 0$, then $\left(\frac{|\phi_n(a)|}{\varepsilon} \right) \to \infty$, it follows that $\left(\sum_n \left[M\left(\frac{|\phi_n(a)|}{\varepsilon} \right) \right]^{r_n} \right)^{1/H} \to \infty$ which is a contradiction. Therefore, $a_{n_m} = 0$ for each m. Finally, we prove that scalar multiplication is continuous. Let λ be any complex number. Then by using (1.5), we have

$$g(\lambda a) \leq (\max(1, |\lambda|^{\sup r_m}))^{1/H} \inf \left\{ s^{r_m/H} : \left(\sum_n \left[M\left(\frac{|\phi_n(a)|}{s} \right) \right]^{r_n} \right)^{1/H} \leq 1, m = 1, 2, \ldots \right\}$$

where $s = \rho/|\lambda|$, which converges to zero as $g(a)$ converges to zero in $|\overline{N}_p|(M, r)$. Now suppose $\lambda_n \to 0$ and a is in $|\overline{N}_p|(M, r)$. For arbitrary $\varepsilon > 0$, let N be a positive integer such that

$$\left(\sum_{n=N+1}^{\infty} \left[m\left(\frac{|\phi_n(a)|}{\rho} \right) \right]^{r_n} \right)^{1/H} < \varepsilon/2 \text{ for some } \rho > 0.$$

Let $0 < |\lambda| < 1$, using convexity of M, we get

$$\sum_{n=N+1}^{\infty} \left[M\left(\frac{|\phi_n(\lambda a)|}{\rho} \right) \right]^{r_n} < \sum_{n=N+1}^{\infty} \left[|\lambda| M\left(\frac{|\phi_n(a)|}{\rho} \right) \right]^{r_n} < (\varepsilon/2)^H.$$

Since M is continuous everywhere in $[0, \infty)$, then

$f(t) = \sum_{n=1}^{N} \left[M\left(\frac{|t\phi_n(a)|}{\rho} \right) \right]^{r_n}$ is continuous at 0. So there is $1 > \delta > 0$ such that $|f(t)| < (\varepsilon/2)^H$ for $0 < t < \delta$.

Let K be such that $|\lambda_k| < \delta$ for $k > K$, then for $k > K$

$$\left(\sum_{n=1}^{N} \left[M\left(\frac{|\lambda_k \phi_n(a)|}{\rho} \right) \right]^{r_n} \right)^{1/H} < \varepsilon/2.$$

Thus $\left(\sum_{n} \left[M\left(\frac{|\lambda_k \phi_n(a)|}{\rho} \right) \right]^{r_n} \right)^{1/H} < \varepsilon$ for $k > K$, so that $g(\lambda a) \to 0$ as $\lambda \to 0$ and so $|\overline{N}_p|(M, r)$ is a topological linear space totally paranormed by g.

3. Inclusion Between $|\overline{N}_p|(M, r)$ Spaces

We now investigate some inclusion relations between $|\overline{N}_p|(M, r)$ spaces.

Theorem 3.1 — *If* $u = (u_k)$ *and* $q = (q_k)$ *are bounded sequences of positive real numbers with* $0 < u_k \le q_k < \infty$ *for each* k, *then for any Orlicz function* M, $|\overline{N}_p|(M, u) \subseteq \overline{N}_p|(M, q)$.

Proof : Let $a \in |\overline{N}_p|(M, u)$. Then there exists some $\rho > 0$ such that $\sum_k \left\{ M\left(\frac{|\phi_k(a)|}{\rho} \right) \right\}^{u_k} < \infty$. This implies that $M\left(\frac{|\phi_k(a)|}{\rho} \right) \le 1$ for all sufficiently large k.

. Since $u_k \le q_k$, $\left\{ M\left(\frac{|\phi_k(a)|}{\rho} \right) \right\}^{q_k} \le \left\{ M\left(\frac{|\phi_k(a)|}{\rho} \right) \right\}^{u_k}$ for large k.

This shows that $a \in |\overline{N}_p|(M, q)$ and completes the proof.

Theorem 3.2 — *If* $r = (r_k)$ *and* $t = (t_k)$ *are bounded sequences of positive real numbers with* $0 < r_k, t_k < \infty$ *and if* $u_k = \min(r_k, t_k)$, $q_k = \max(r_k, t_k)$, *then for any Orlicz function* M, $|\overline{N}_p|(M, u) = |\overline{N}_p|(M, r) \cap |\overline{N}_p|(M, t)$ *and* $|\overline{N}_p|(M, q) = g$, *where* g *is the subspace of* w *(the space of all complex sequences) generated by* $|\overline{N}_p|(M, r) \cup |\overline{N}_p|(M, t)$.

Proof : It follows from Theorem 3.1 that

$$|\overline{N}_p|(M, u) \subseteq |\overline{N}_p|(M, r) \cap |\overline{N}_p|(M, t) \text{ and that } g \subseteq |\overline{N}_p|(M, q).$$

For any complex λ, $|\lambda|^{u_k} \le \max(|\lambda|^{r_k}, |\lambda|^{t_k})$, thus $|\overline{N}_p|(M, r) \cap |\overline{N}_p|(M, t) \subseteq |\overline{N}_p|(M, u)$.

Let $A = \{k : r_k \ge t_k\}$ and $B = \{k : r_k < t_k\}$.

If $a = (a_k) \in |\overline{N}_p|(M, q)$, we write

$$b_k = a_k\ (k \in A) \text{ and } b_k = 0\ (k \in B); \text{ and}$$

$$c_k = 0\ (k \in A) \text{ and } c_k = a_k\ (k \in B).$$

Then since $a = (a_k) \in |\overline{N}_p|(M, q)$, there exists some $\rho > 0$ such that

$$\sum_k \left\{ M\left(\frac{|\phi_k(a)|}{\rho} \right) \right\}^{q_k} < \infty.$$

Now,
$$\sum_k \left\{ M\left(\frac{|\phi_k(b)|}{\rho} \right) \right\}^{r_k} = \sum_{k \in A} + \sum_{k \in B} = \sum_{k \in A} \left\{ M\left(\frac{|\phi_k(a)|}{\rho} \right) \right\}^{q_k} < \infty$$

and so $\quad b \in |\overline{N}_p|(M, r) \subseteq g.$

Similarly, $c \in |\overline{N}_p|(M, t) \subseteq g$.

Thus, $a = b + c \in g$. We have proved that $|\overline{N}_p|(M, q) \subseteq g$, which gives the required result.

Corollary 3.2 — The three conditions $|\overline{N}_p|(M, r) \subseteq |\overline{N}_p|(M, t)$, $|\overline{N}_p|(M, r) \quad |\overline{N}_p|(M, u) = |\overline{N}_p|(M, r)$ and

$|\overline{N}_p|(M, t) = |\overline{N}_p|(M, q)$ are equivalent.

Corollary 3.3 — $|\overline{N}_p|(M, r) = |\overline{N}_p|(M, t)$ if and only if $|\overline{N}_p|(M, u) = |\overline{N}_p|(M, q)$.

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