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HOW TO CALCULATE THE CONTENT FOR A GIVEN REALIZABLE FUZZY MATRIX*

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A $n \times n$ fuzzy matrix A is called realizable if there exists a $n \times t$ fuzzy matrix B such that $A = B \odot B^T$, where "$\odot$" is the max-min composition, B^T denotes the transpose of B. Let $r(A) = \min \{t : A = B \odot B^T$ for $B \in L^{n \times t}\}$ (where $L = [0, 1]$), then $r(A)$ is called the content of A. From 1982 to now, how to calculate $r(A)$ for a given $n \times n$ realizable fuzzy matrix A is not attacked. In this pape,r we propose an algorithm which can calculate $r(A)$ for a given $n \times n$ realizable fuzzy matrix A and produce a $n \times r(A)$ fuzzy matrix B such that $A = B \odot B^T$, within $1^{n^2} + 2^{n^2} + \ldots + [r(A)]^{n^2}$ steps.

Key Words : Fuzzy Matrix (Relations) : Realizable Fuzzy Matrix; Content of a Realizable Fuzzy Matrix; Algorithm of the Content

§ 1. INTRODUCTION

By denoting with $L = ([0, 1], \leq)$, where "$\leq$" is the usual ordering of real numbers, $I_n = \{1, 2, \ldots, n\}$ the set of first "n" natural number, and

$$L^{n \times m} = \{A = (A_{ij})_{n \times m} = (A_{ij}) : A_{ij} \in [0, 1], i \in I_n, j \in I_m\}.$$

Usually, any element $A = (A_{ij}) \in L^{n \times m}$ is called a $n \times m$ fuzzy matrix.

Recalling that the total ordering "$\leq$" of L induces a partial ordering in $L^{n \times m}$ defined as

$$A \leq B \text{ if and only if } A_{ij} \leq B_{ij}$$

for any $i \in I_n$ and $j \in I_m$, where $A = (A_{ij})$ and $B = (B_{ij}) \in L^{n \times m}$ (see [1]).

Hence, $A \nleqq B$ means that $A_{i_0 j_0} > B_{i_0 j_0}$ for some $i_0 \in I_n, j_0 \in I_m$.

Let $A = (A_{ij}) \in L^{n \times m}$ and $B = (B_{ij}) \in L^{m \times p}$, then the max-min composition of A and B, denoted by $A \odot B = ((A \odot B)_{ij}) \in L^{n \times p}$, is defined as follows :

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$$(A \odot B)_{ij} = \bigvee_{k=1}^{m} (A_{ik} \wedge B_{kj})$$

for any $i \in I_n$ and $j \in I_p$, where "$\wedge$" and "$\vee$" denote, respectively, the usual maximum and minimum operator of L (see [4]). And the transpose of A, denoted by A^T is defined by $A^T \in L^{m \times n}$ and

$$(A^T)_{ij} = A_{ji}$$

for any $i \in I_m$ and $j \in I_n$ (see [4]).

Let $A = (A_{ij})$ and $B = (B_{ij}) \in L^{n \times m}$, then the union $A \vee B = ((A \vee B)_{ij}) \in L^{n \times m}$ of A and B is defined as $(A \vee B)_{ij} = A_{ij} \vee B_{ij}$ for any $i \in I_n, j \in I_m$.

In 1982, Liu (see [2]) gave the concept of a realizable fuzzy matrix as follows :

Definition 1.1 (Liu ([2, Definition 2.1]) — A fuzzy matrix $A \in L^{n \times n}$ is called realizable if there exists a $n \times t$ fuzzy matrix B such that $A = B \odot B^T$. Further, let

$$r(A) = \min \{t : A = B \odot B^T \text{ for } B \in L^{n \times t}\},$$

then $r(A)$ is called the content for the realizable fuzzy matrix A.

Remark 1.1 : In Definition 1.1, the $n \times t$ fuzzy matrix B is not generally unique.

Example 1.1 — Let

$$A = B_1 = \begin{pmatrix} 0.3 & 0.2 \\ 0.2 & 0.2 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0.3 \\ 0.2 \end{pmatrix},$$

then we can check that $A = B_1 \odot (B_1)^T$, in this case $t = 2$. It is clear that $A = B_2 \odot (B_2)^T$, in this case $t = 1$.

For a fuzzy matrix $A \in L^{n \times n}$, when we consider its realizability, there are two key problems which need to be attacked :

Problem 1 — What are the conditions under which a fuzzy matrix $A \in L^{n \times n}$ is realizable?

Problem 2 — For a given realizable fuzzy matrix $A \in L^{n \times n}$, how to calculate its content $r(A)$?

In 1984, Wang (see [5]) attacked Problem 1, he gave the following proposition :

Proposition 1.1 (Wang [5, Theorem 2.1]) — Let $A = (A_{ij}) \in L^{n \times n}$, then A is realizable if and only if

$$A_{ii} \geq A_{ij} = A_{ji}$$

for any $\quad i, j \in I_n$.

Many authors have made many scientific researches in order to solve Problem 2 (the readers can see [2, 3, 5-8] for further details), Liu's results (see [3]) are the deepest till now. Let

$$r_n(L) = min \{r : \text{for all realizable matrices } A \in L^{n \times n}, r(A) \leq r\}.$$

Liu (see [3]) obtained the followings :

Proposition 1.2 (Liu [3, Theorem 3.1]) — For $n = 1, 2, 3$, $r_n(L) = n$.

Proposition 1.3 (Liu [3, Theorem 3.2]) — For $n \geq 4$, $r_n(L) = [n^2/4]$, where $[n^2/4]$ is the integral part of $n^2/4$.

It is obvious that the results above (which include all the results given by [2, 3, 5-8]) can only given an estimative value of the content for a given realizable fuzzy matrix $A \in L^{n \times n}$. Hence, Problem 2 is still an open problem.

In this paper, we propose an algorithm which not only can calculate the content $r(A)$ for a given realizable fuzzy matrix $A \in L^{n \times n}$ but also produce a $n \times r(A)$ fuzzy matrix B such that $A = B \odot B^T$.

§ 2. PRELIMINARIES

Let $A = (A_{ij}) \in L^{n \times n}$ and t be a given positive integer. For every $h, k \in I_n$ and $q \in I_t$, we define a $n \times t$ fuzzy matrix $M^{(h,q,k)} = ([M)^{h,q,k)}]_{ij})$ as follows :

$$[M^{(h,q,k)}]_{ij} = \begin{cases} A_{hk} & \text{if } i = h, j = q, \\ A_{kh} & \text{if } i = k, j = q, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{... (2.1)}$$

where $i \in I_n$ and $j \in I_t$. Let

$$M^{T(h,q,k)} = ([M^{T(h,q,k)}]_{ij}) = [M^{(h,q,k)}]^T, \qquad \text{... (2.2)}$$

then $M^{T(h,q,k)} \in L^{t \times n}$.

The example below illustrates the above definition.

Example 2.1 — Let $t = 2$ and

$$A = \begin{bmatrix} 1.0 & 0.4 & 0.0 & 0.8 \\ 0.5 & 0.7 & 0.1 & 0.2 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.1 & 0.9 \end{bmatrix},$$

then

$$M^{(1,2,3)} = \begin{pmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.3 \\ 0.0 & 0.0 \end{pmatrix}, \quad M^{(4,1,3)} = \begin{pmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \\ 0.5 & 0.0 \\ 0.1 & 0.0 \end{pmatrix} \text{ and } M^{(2,1,2)} = \begin{pmatrix} 0.0 & 0.0 \\ 0.7 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \end{pmatrix}.$$

Remark 2.1 : It is clear that the number of distinct $n \times t$ fuzzy matrices defined by (2.1) is not more than tn^2.

We have the following :

Proposition 2.1 —Let $A = (A_{ij}) \in L^{n \times n}$ be a realizable fuzzy matrix, $M^{(h,q,k)} = ([M^{(h,q,k)}]_{ij}) \in L^{n \times t}$ defined as (2.1) for any $h, k \in I_n$ and $q \in I_t$, then

$$\bigvee_{p=1}^{t} \left\{ [M^{(h,q,k)}]_{hp} \wedge [M^{T(h,q,k)}]_{pk} \right\} = A_{hk}.$$

PROOF : We have $A_{ij} = A_{ji}$ for any $i, j \in I_n$ by Proposition 1.1 since A is realizable. By (2.1), we deduce that

$$\bigvee_{p=1}^{t} \left\{ [M^{(h,q,k)}]_{hp} \wedge [M^{T(h,q,k)}]_{pk} \right\} = [M^{(h,q,k)}]_{hq} \wedge [M^{T(h,q,k)}]_{qk}$$

$$= A_{hk} \wedge A_{kh}$$

$$= A_{hk}$$

for any $h, k \in I_n$.

§ 3. THE ALGORITHM OF THE CONTENT FOR A GIVEN REALIZABLE FUZZY MATRIX

By putting

$$u = k + (h - 1)\, n \qquad \text{... (3.1)}$$

for any $h, k \in I_n$. Thus $1 \le u \le n^2$, and for a given positive integer t arbitrarily choosing the index i_u, $1 \le i_u \le t$, with u given from (3.1), we consider the $n \times t$ fuzzy matrix $M = (M_{ij})$ defined as

$$M = \bigvee_{u=1}^{n^2} M^{(u)}, \qquad \text{... (3.2)}$$

where $M^{(u)} = M^{(h, i_u, k)} \in L^{n \times t}$ with $M^{(h, i_u, k)}$ defined as (2.1). Then we have the following :

Proposition 3.1 — If A is a realizable fuzzy matrix, then $M \odot M^T \ge A$, where M defined by (3.2).

PROOF : Let M be a $n \times t$ fuzzy matrix defined by (3.2), then

$$M_{ij} \ge [M^{(u)}]_{ij}$$

for any $i \in I_n, j \in I_t$ and $u \in I_{n^2}$. By (2.2) and Proposition 2.1, we have

$$(M \odot M^T)_{hk} = \bigvee_{p=1}^{t} [M_{hp} \wedge (M^T)_{pk}]$$

$$\ge \bigvee_{p=1}^{t} \{[M^{(u)}]_{hp} \wedge \{[M^{(u)}]^T\}_{pk}\}$$

$$= \bigvee_{p=1}^{t} \{[M^{(h, i_u, k)}]_{hp} \wedge [M^{T(h, i_u, k)}]_{pk}\}$$

$$= A_{hk}$$

for any $h, k \in I_n$. This completes the proof.

We give an explicit example to explain formula (3.2) and Proposition 3.1 below.

Example 3.1 — Let $t = 2$ and

$$A = \begin{bmatrix} 1.0 & 0.5 & 0.0 & 0.4 \\ 0.5 & 0.7 & 0.1 & 0.6 \\ 0.0 & 0.1 & 0.1 & 0.1 \\ 0.4 & 0.6 & 0.1 & 0.6 \end{bmatrix}.$$

According to Proposition 1.1, A is a realizable fuzzy matrix. Here we only give two fuzzy matrices according to (3.2) as follows :

$$M_1 = M^{(1,1,1)} \vee M^{(1,1,2)} \vee M^{(1,1,3)} \vee M^{(1,1,4)}$$

$$\vee M^{(2,1,1)} \vee M^{(2,2,2)} \vee M^{(2,2,3)} \vee M^{(2,2,4)}$$

$$\vee M^{(3,1,1)} \vee M^{(3,2,2)} \vee M^{(3,2,3)} \vee M^{(3,2,4)}$$

$$\vee M^{(4,1,1)} \vee M^{(4,2,2)} \vee M^{(4,2,3)} \vee M^{(4,2,4)}$$

$$= \begin{bmatrix} 1.0 & 0.0 \\ 0.5 & 0.7 \\ 0.0 & 0.1 \\ 0.4 & 0.6 \end{bmatrix},$$

and

$$M_2 = M^{(1,1,1)} \vee M^{(1,1,2)} \vee M^{(1,1,3)} \vee M^{(1,1,4)}$$

$$\vee M^{(2,1,1)} \vee M^{(2,2,2)} \vee M^{(2,2,3)} \vee M^{(2,2,4)}$$

$$\vee M^{(3,1,1)} \vee M^{(3,2,2)} \vee M^{(3,2,3)} \vee M^{(3,2,4)}$$

$$\vee M^{(4,1,1)} \vee M^{(4,2,2)} \vee M^{(4,2,3)} \vee M^{(4,1,4)}$$

$$= \begin{bmatrix} 1.0 & 0.0 \\ 0.5 & 0.7 \\ 0.0 & 0.1 \\ 0.6 & 0.6 \end{bmatrix}.$$

Hence, we deduce that

$$M_1 \odot (M_1)^T = A \text{ and } M_2 \odot (M_2)^T = \begin{bmatrix} 1.0 & 0.5 & 0.0 & 0.6 \\ 0.5 & 0.7 & 0.1 & 0.6 \\ 0.0 & 0.1 & 0.1 & 0.1 \\ 0.6 & 0.6 & 0.1 & 0.6 \end{bmatrix} \geq A.$$

Remark 3.1 : It is easy to see that the number of distinct $n \times t$ fuzzy matrices M defined by (3.2) is not more than t^{n^2} for a given $n \times n$ fuzzy matrix A.

We need the lemma below :

Lemma 3.1 (Sanchez[4]) — Let A_1 and $A_2 \in L^{n \times m}, B_1$ and $B_2 \in L^{m \times p}$. If $A_1 \leq A_2$ $(B_1 \leq B_2)$, then $A_1 \odot B_1 \leq A_2 \odot B_2$.

Let $A \in L^{n \times n}$,

$$\mathcal{B} = \{B \in L^{n \times t} : B \odot B^T = A\},$$

and

$$\mathcal{B}_0 = \{M \in L^{n \times t} \text{ and } M \text{ defined by (3.2)} : M \odot M^T = A\}.$$

Then we have the following :

Proposition 3.2 — $\mathcal{B}_0 \neq \emptyset$ if and only if $\mathcal{B} \neq \emptyset$.

PROOF : We only prove the non-trival implication since $\mathcal{B}_0 \subseteq \mathcal{B}$. Let $B = (B_{ij}) \in \mathcal{B}$, then $B \odot B^T = A$. Hence, for every $h, k \in I_n$ there exists an index $q \in I_t$ such that

$$A_{hk} = B_{hq} \wedge (B^T)_{qk}$$

$$= \bigvee_{p=1}^{t} [B_{hp} \wedge (B^T)_{pk}]. \quad \text{... (3.3)}$$

This implies either

$$B_{hq} = A_{hk} \leq (B^T)_{qk} \quad \text{... (3.4)}$$

or

$$(B^T)_{qk} = A_{hk} \leq B_{hq}. \quad \text{... (3.5)}$$

If (3.4) holds, then we obtain

$$B_{hq} = A_{hk} = [M^{(h,q,k)}]_{hq} \text{ and } (B^T)_{qk} \geq A_{hk} = A_{kh} = [M^{T(h,q,k)}]_{qk}.$$

If (3.5) holds, then we deduce that

$$B_{hq} \geq A_{hk} = [M^{(h,q,k)}]_{hq} \text{ and } (B^T)_{qk} = A_{hk} = A_{kh} = [M^{T(h,q,k)}]_{qk}.$$

Since

$$B_{ij} \geq 0 = [M^{(h, q, k)}]_{ij}$$

for any pair $(i, j) \in I_n \times I_t - \{(h, q), (k, q)\}$ in both cases, then for every $h, k \in I_n$, we can determine a $n \times t$ fuzzy matrix $M^{(h, q, k)}$ with $q \in I_t$, determined as (3.3), such that

$$M^{(h, q, k)} \leq B.$$

By putting $i_u = q$ with u defined from (3.1), we have

$$M^{(h, i_u, k)} \leq B.$$

Thus we have determined a n^2-tuple $(i_1, i_2, \ldots, i_{n^2})$ such that $n \times t$ fuzzy matrix M, defined from (3.2), satisfies $M \leq B$. By Proposition 3.1 and Lemma 3.1, we obtain that

$$A \leq M \odot M^T \leq B \odot B^T = A.$$

Therefore, $M \odot M^T = A$, i.e., $M \in \mathcal{B}_0$. This concludes the proof.

Remark 3.2 : In account of Proposition 3.2, $A \in L^{n \times n}$ is realizable if and only if there exists a $n \times t$ fuzzy matrix M, given from (3.2), belonging to $\mathcal{B}_0$.

Following Sanchez (see [4]), we recall the binary operation $\alpha : L^2 \to L$ defined

$$x \alpha y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y, \end{cases} \quad \ldots (3.6)$$

where $x, y \in L$ and the α-composition $A \,\alpha B = ((A \,\alpha B)_{ij}) \in L^{n \times p}$ of $A = (A_{ij}) \in L^{n \times m}$ and $B = (B_{ij}) \in L^{m \times p}$, given by

$$(A \,\alpha B)_{ij} = \bigwedge_{k=1}^{m} (A_{ik} \,\alpha B_{kj}),$$

where $i \in I_n, j \in I_p$.

Lemma 3.2 (Sanchez [4, Theorem 3]) — Let $A \in L^{n \times n}$ and $B \in L^{n \times t}$, then

$$B \odot (B^T \alpha A) \leq A.$$

Lemma 3.3 (Sanchez [4, Theorem 5]) — Let $A \in L^{n \times n}$ and $B \in L^{n \times t}$, and

$$\mathcal{H} = \{C \in L^{t \times n} : B \odot C = A\},$$

then $\mathcal{H} \neq \emptyset$ if and only if $B^T \alpha A \in \mathcal{H}$. Further, $B^T \alpha A$ is the greatest element in $\mathcal{H}$.

Proposition 3.3 — $A \in L^{n \times n}$ is realizable if and only if there exists a $n \times t$ fuzzy matrix M, given from (3.2), such that $M^T \leq M^T \alpha A$.

PROOF : If $A \in L^{n\times n}$ is realizable, then by Remark 3.2 there exists an element $M \in \mathcal{B}_0$ such that

$$A = M \odot M^T.$$

Further by Lemma 3.3, we have that

$$M^T \leq M^T \alpha A.$$

Vice versa, using Lemma 3.2, Lemma 3.1 and Proposition 3.1, we have that

$$A \leq M \odot M^T \leq M \odot (M^T \alpha A) \leq A,$$

which implies $A = M \odot M^T$. So that A is realizable. The proof is completed.

By Proposition 3.2, 3.3 and Definition 1.1, we can given algorithm, as follows, to calculate the content $r(A)$ for a given $n \times n$ realizable fuzzy matrix A, and produce a $n \times r(A)$ fuzzy matrix M such that $A = M \odot M^T$.

ALGORITHM 3.1 — t — the column index of the fuzzy matrix M.

l — the number of times of building the $n \times t$ fuzzy matrix M, corresponding to t.

$M^{(h, i_u, k)}$ — then $n \times t$ fuzzy matrix defined by (2.1), corresponding to t.

Step 1. $t = 1$

Step 2. $l = 1$

Step 3. Builds $M = \bigvee_{\substack{1 \leq h \leq n \\ 1 \leq k \leq n}} M^{(h, i_{k+(h-1)n}, k)}$, where

$$M^{(h, i_{k+(h-1)n}, k)} \in \{M^{(h, i_{k+(h-1)n}, k)} : i_{k+(h-1)n} \in I_t\} \text{ for every } h, k \in I_n.$$

Step 4. Compares the M^T with $M^T \alpha A$.

Step 5. When $M^T \not\leq M^T \alpha A$: $l = l + 1$ go to Step 3 if $l \neq t^{n^2}$, otherwise $t = t + 1$ go to Step 2.

Step 6. $r(A) = t$.

Step 7. End.

The proposition below is immediately from Proposition 3.2, 3.3 and Definition 1.1, we omit its proof.

Proposition 3.4 — Let $A \in L^{n\times n}$ be a realizable fuzzy matrix, then the Algorithm 3.1 can find a $n \times r(A)$ fuzzy matrix M such that $A = M \odot M^T$ within $1^{n^2} + 2^{n^2} + \ldots + [r(A)]^{n^2}$ steps.

Proposition 3.5 — Let $A \in L^{n\times n}$, $B \in L^{n\times t}$ and $C \in L^{t\times n}$. If $C \not\leq B^T \alpha A$, then $C \vee Q \not\leq (B \vee P)^T \alpha A$ for any $P \in L^{n\times t}$ and $Q \in L^{t\times n}$.

PROOF : By $C \not\leq B^T \alpha A$, we have

$$C_{i_0j_0} > \bigwedge_{k=1}^{n} [(B^T)_{i_0k}\, \alpha A_{kj_0}]$$

$$= \bigwedge_{k=1}^{n} (B_{ki_0}\, \alpha A_{kj_0})$$

for some $i_0 \in I_t, j_0 \in I_n$. Thus there exists an index $k_0 \in I_n$ such that

$$C_{i_0j_0} > B_{k_0i_0}\, \alpha A_{k_0j_0}.$$

Then by (3.6), we have that

$$B_{k_0i_0} > A_{k_0j_0} \qquad \text{... (3.7)}$$

and

$$C_{i_0j_0} > A_{k_0j_0} \qquad \text{... (3.8)}$$

since $B_{k_0i_0}\, \alpha A_{k_0j_0} < 1$. Suppose that there exist $P \in L^{n\times t}$ and $Q \in L^{t\times n}$ such that

$$C \vee Q \le (B \vee P)^T\, \alpha A,$$

which implies that

$$(C \vee Q)_{i_0j_0} \le [(B \vee P)^T\, \alpha A]_{i_0j_0}.$$

That is

$$C_{i_0j_0} \vee Q_{i_0j_0} \le \bigwedge_{k=1}^{n} [(B \vee P)_{ki_0}\, \alpha A_{kj_0}].$$

So that

$$C_{i_0j_0} \vee Q_{i_0j_0} \le (B \vee P)_{k_0\,i_0}\, \alpha A_{k_0j_0}.$$

Because

$$(B \vee P)_{k_0i_0} = B_{k_0i_0} \vee P_{k_0i_0}$$

$$> A_{k_0j_0}$$

by (3.7), we have

$$C_{i_0j_0} \vee Q_{i_0j_0} \le A_{k_0j_0},$$

i.e., $C_{i_0 j_0} \leq A_{k_0 j_0}$. This contradicts (3.8). The proof is completed.

Proposition 3.6 — Let $A \in L^{n \times n}, B_i \in L^{n \times t}$ and $C_i \in L^{t \times n}, i \in I_2$. If $C_1 \vee C_2 \leq (B_1 \vee B_2)^T \alpha A$, then $C_i \leq (B_i)^T \alpha A$ for $i \in I_2$.

PROOF : Suppose that $C_1 \underline{\not\leq} B_1^T \alpha A$; then by Proposition 3.5, we have that $C_1 \vee C_2 \underline{\not\leq} (B_1 \vee B_2)^T \alpha A$. This is in contradiction with the hypothesis.

Proposition 3.7 — Let $A \in L^{n \times n}$ and $t \in I_n$. For fixed $h, k \in I_n$ if for every M defined by (3.2),

$$M^{T(h, i_u, k)} \underline{\not\leq} [M^{T, (h, i_u, k)}] \alpha A$$

for all $i_u \in I_t$, Then $A \neq B \odot B^T$ for all $B \in L^{n \times t}$.

PROOF : If $A = B \odot B^T$ for some $B \in L^{n \times t}$, then by Definition 1.1 and Proposition 3.3, there exists a $n \times t$ fuzzy matrix M, given from (3.2), such that $M^T \leq M^T \alpha A$. Thus by (3.2) and Proposition 3.6, we have that

$$[M^{(u)}]^T \leq [M^{(u)}]^T \alpha A$$

for all $u \in I_{n^2}$ with

$$M = \bigvee_{u=1}^{n^2} M^{(u)}.$$

So that

$$[M^{(u)}]^T \leq [M^{(u)}]^T \alpha A,$$

where u is determined by the fixed $h, k \in I_n$ and using (3.1). Furthermore by $M^{(u)} = M^{(h, i_u, k)}$, we have that

$$[M^{(h, i_u, k)}]^T \leq [M^{(h, i_u, k)}]^T \alpha A,$$

i.e.,

$$M^{T(h, i_u, k)} \leq [M^{T(h, i_u, k)}] \alpha A$$

for the fixed $h, k \in I_n$. This contradicts the hypothesis. The proof is completed.

In account of Proposition 3.7, further we can give a flowchart of the Algorithm 3.1 as Fig. 3.1.

Remark 3.3 : The index t in the algorithm as Fig. 3.1 denotes all the current fuzzy matrices $M^{(u)}$ in Fig. 3.1, corresponding to the index t, are $n \times t$ fuzzy matrices.

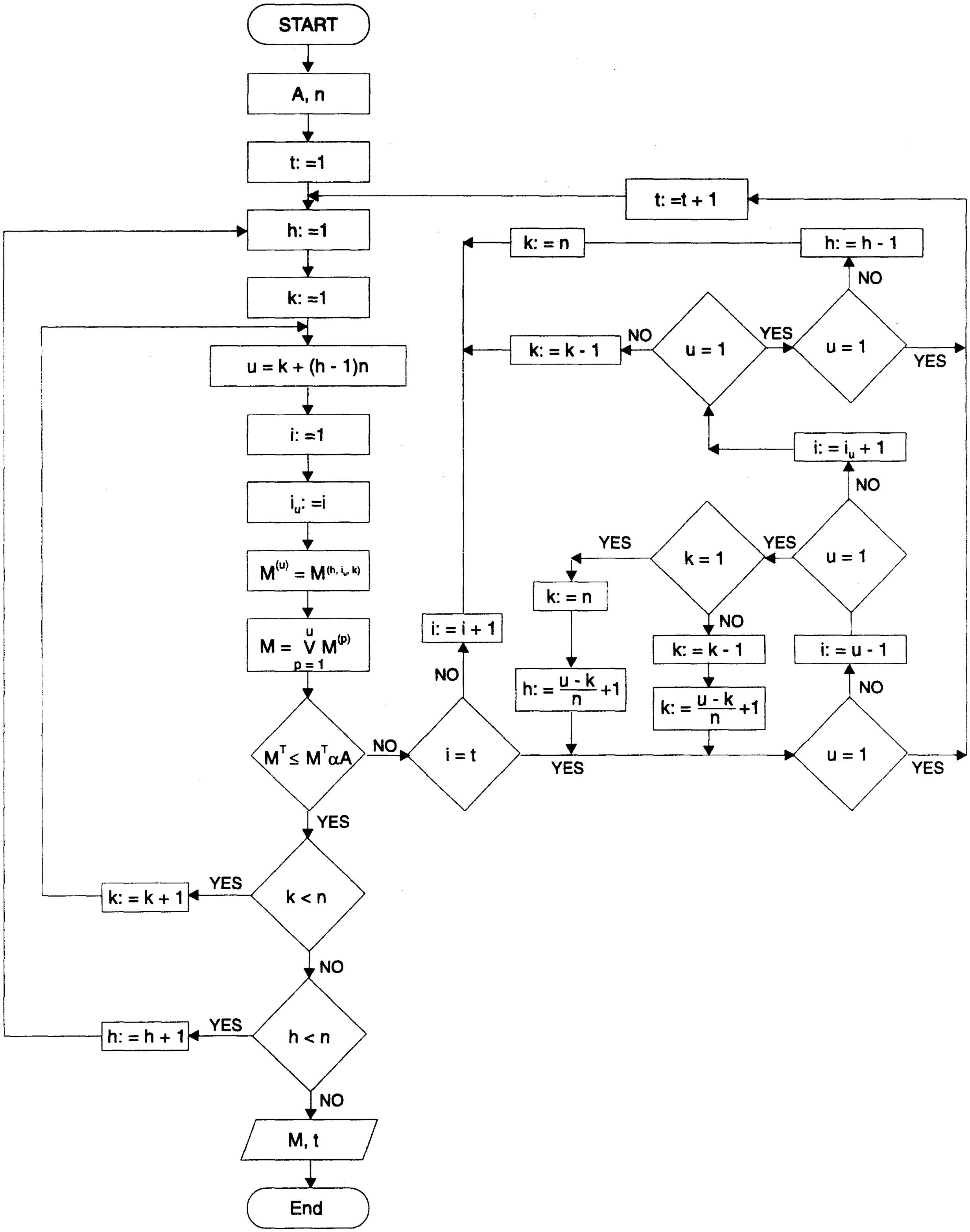

FIG. 1. Flowchart of the algorithm of the content $r(A)$ for a given realizable fuzzy matrix $A \in L^{n\times n}$

Remark 3.4 : There is a trivial algorithm whose computing volume is $\left[\frac{(1+n)n}{2}\right]^{n\times 1}$ + $\left[\frac{(1+n)n}{2}\right]^{n\times 2} n\times r(A)$ +...+ $\left[\frac{(1+n)n}{2}\right]^{n\times n} r(A)$: We start from t = 1, try to place all the entries of $A \in L^{n\times n}$, in all possible permutations into a matrix $B \in L^{n\times t}$. Then the first t on which there is a matrix $B \in L^{n\times t}$ through permuting as above, such that $A = B \odot B^T$ must be $r(A)$.

Proposition 3.8 — Let $A \in L^{n\times n}$ be a realizable fuzzy matrix. If the number of distinct entries in A is greater than $n \times t$, then $r(A) \geq t+1$.

PROOF : If $r(A) \leq t$, then by Definition 1.1, there is a fuzzy matrix $B \in L^{n\times r(A)}$ such that $A = B \odot B^T$. So that the number of distinct entries in A isn't greater than $n \times r(A)$ since $\odot$ is a max-min operator. This contradicts the hypothesis.

Proposition 3.9 — Let $A = (a_{ij})_{n\times n}$ be a realizable fuzzy matrix. If there are t entries $a_{i_1 i_1}, \ldots, a_{i_t i_t}$, in A such that $a_{i_j i_j} > a_{i_j k}$ for every $j \in I_t$ and any $k \in I_n$, then $r(A) \geq t$.

PROOF : By Proposition 3.2, there exists $M = (M_{ij})_{n\times r(A)}$ such that $A = M \odot M^T$. Hence

$$a_{i_j i_j} = \bigvee_{k=1}^{r(A)} (M_{i_j k} \wedge M_{i_j k}) = \bigvee_{k=1}^{(A)} M_{i_j k}$$

for every $j \in I_t$. So that there exists an $k_{i_j} \in I_{r(A)}$ such that $a_{i_j i_j} = M_{i_j k_{i_j}}$. If $r(A) < t$, then there exist at least two entries, which is equal, among $k_{i_1}, k_{i_2}, \ldots, k_{i_t}$. Without loss of general, we suppose $k_{i_{i_0}} = k_{i_{j_0}}$, $i_0 \neq j_0$, $1 \leq i_0, j_0 \leq t$. Then

$$a_{i_{i_0} i_{i_0}} \wedge a_{i_{j_0} i_{j_0}} = M_{i_{i_0} k_{i_{i_0}}} \wedge M_{i_{j_0} k_{i_{j_0}}}$$

$$\leq \bigvee_{k=1}^{r(A)} (M_{ij_0 k} \wedge M_{ij_0 k})$$

$$= a_{i_{i_0} i_{j_0}}.$$

This contradicts the hypothesis.

Remark 3.5 : By Proposition 3.8, 3.9, if the number of distinct entries in A is greater than $n \times (p-1)$, and there are q entries $a_{i_1 i_1}, \ldots, a_{i_q i_q}$ in A such that $a_{i_j i_j} > a_{i_j k}$ for every $j \in I_q$ and $k \in I_n$, then we can let the initial value of t be max $\{p, q\}$ in Figure 3.1, in order to speed up the computation of $r(A)$.

Example 3.2 — Let

$$A = \begin{bmatrix} 0.6 & 0.4 & 0.3 \\ 0.4 & 0.5 & 0.0 \\ 0.3 & 0.0 & 0.3 \end{bmatrix}.$$

Consider (1) is A a realizable fuzzy matrix?

(2) If A is a realizable fuzzy matrix, then $r(A) = ?$

Solution — (1) By Proposition 1.1, we know that A is realizable.

(2) Applying the algorithm of Fig. 3.1, we can build a fuzzy matrix $M = \begin{bmatrix} 0.6 & 0.4 \\ 0.0 & 0.5 \\ 0.3 & 0.0 \end{bmatrix}$ such that $A = M \odot M^T$, and we also know that $r(A) = 2$ by computing.

REFERENCES

1. J. A. Goguen, *J. Math. Anal. Appl.*, **18** (1967), 145-174.
2. Wangjin Liu, *J. Fuzzy Math.*, **1** (1982), 69-76.
3. Xuecheng Liu, *Fuzzy Sets Syst.*, **80** (1996), 257-259.
4. E. Sanchez, *Info. Control*, **30** (1976), 38-48.
5. Mingxin Wang, *J. Fuzzy Math.*, **1** (1984), 51-58.
6. Geping Wang, *J. Fuzzy Math.*, **2** (1985), 65-74.
7. Yandong Yu, *J. Fuzzy Math.*, **1** (1984), 15-20.
8. Duo Zhao, *J. Fuzzy Math.*, **1** (1985), 1-8.