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LAMINAR FLOW IN A UNIFORMLY POROUS PIPE

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The steady laminar flow of a viscous incompressible fluid through a uniformly porous tube is considered. The similarity transformation is used to reduce the governing equations into nonlinear ordinary differential equation. The solution procedure includes application of long series analysis with polynomial coefficients. Thirty five effective terms in the series representing skin friction (shear stress at wall) and velocity profile reveal qualitative features which are comparable to pure numerical results. To this order the number of universal coefficients generated are 1260 in each case. After identifying and estimating the location and nature of nearest singularity restricting the convergence of the series, the series are recast into new forms using Euler transformation whose region of validity increases up to moderately large values of Reynolds numbers. The present analysis enables in extending region of validity from $-241 \le R \le -24$ (numerical) to $-300 \le R \le 1$ in predicting shear stress at the surface. A complete description of the solutions is presented and values of skin friction and velocity profiles are obtained.

Key Words : **Laminar Flow; Uniformly Porous Pipe; Viscous Incompressible Fluid; Nonlinar Ordinary Differential Equation; Skin Friction; Transpiration Cooling; Gaseous Diffusion Technology; Pade approximates; Euler transformation.**

NOMENCLATURE

u = *axial velocity component.*

ρ = *density of fluid.*

$\eta = \left[\dfrac{r}{r_w}\right]^2$ = *dimensionless radial variable.*

p = *pressure.*

r_w = *radius of pipe.*

ε_0 = *radius of convergence of series (3.12)*

v = *radial velocity component.*

r = *radial coordinate.*

v_w = *velocity of fluid through porous walls.*

ρ = *density of fluid.*

μ = *viscosity of fluid.*

$R = \dfrac{\rho v_w r_w}{\mu}$ = *Reynolds number.*

ν = *Coefficient of Kinematic viscocity.*

1. Introduction

Transpiration cooling, gaseous diffusion technology, cooling of rocket, etc, are the examples of diffusion of fluids at walls. This includes flow through cylindrical pipes and channels with mass transfer at the wall. The process commonly used for decreasing drag and increasing lift of airplane wings is boundary layer control. In this case, boundary flow is sucked through the surface of the wing to a duct.

For a general boundary shape, and for a prescribed suction or injection, it is necessary to solve system of partial differential equations. In the case of channel and pipe flows, Berman[1] has shown that, for constant suction or injection at the walls, the problem may be reduced to solving a single ordinary nonlinear differential equation.

The solution of the problem of laminar flow in a uniformly porous channel has been discussed by various authors. The numerical work on the problem of flow in a porous pipe, by Berman[2] and White[3] revealed a multiplicity of solutions for certain ranges of suction velocity, and established that at a certain critical suction rate the velocity gradient at the wall becomes zero. Berman[2] pointed out that for the circular tube geometry suction could induce a transition to turbulence. White[3] and Terrill[4] have shown that the solutions exhibit a marked instability with no solutions obtainable for some values of the suction. No dual solutions were obtained by Yuan and Finkelstein[5] by using a regular perturbation technique.

For simple geometries the semi-numerical analytical methods proposed here provide accurate results and have advantages over pure numerical results. A single computer run yields the solution for a large range of expansion quantitity. Once the convergence of the series is guaranteed the results can be obtained upto any desired accuracy. Aslo any derived quantity can be obtained easily whereas in the case of numerical schemes it requires elaborate additional schemes. In addition, this method reveals the analytic structure of the solution which is absent in numerical solutions.

The method used consists of three steps. First the set of perturbation equations are programmed for the solution on digital computer so that a large number of terms can be generated. Second the coefficients of the series are utilised to identify the location and nature of singularities limiting the range of applicability of the series. Based on this knowledge, the final step is to recast the series using one or a combination of devices such as, Euler transformation, Pade' approximants, series reversion etc, to improve the region of validity of the series. Recently, this technique is used successfully bu the authors[6&7] in the analysis of lubrication of long porous slider and Laminar flow through porous walls of different permeability.

In this paper, a solution valid for large injection Reynolds number and small suction Reynolds number is presented. The point where the skin-friction vanishes for suction Reynolds number is located by using series analysis.

For the case of suction at both walls the coefficients of the series representing the skin friction and axial velocity profiles are having fixed sign pattern. For large injection the sign pattern of the coefficients of the above two series are alternating. The process of improvement involves the use of Euler transformation and application of Pade' approximants.

2. Formulation of the Problem

Consider a steady, incompressible laminar flow through a pipe of circular cross section with porous walls. The flow is axisymmetric with negligible body forces. The uniform porosity is simulated by prescribing a constant normal wall velocity v_w. The schematic diagram of the problem is shown in Fig. 1.

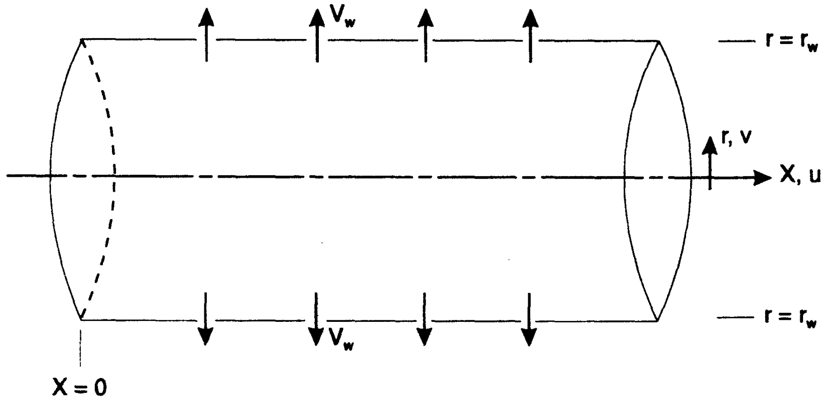

FIG 1. Schematic diagram of the flow.

The Navier-Stokes equations of the flow region are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{v}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial x}\left(r\frac{\partial u}{\partial x}\right)\right] \qquad \text{... (2.1)}$$

and

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{v}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial}{\partial x}\left(r\frac{\partial v}{\partial x}\right)-\frac{v}{r}\right]. \qquad \text{... (2.2)}$$

The equation of continuity is

$$\frac{\partial}{\partial x}(ru)+\frac{\partial}{\partial r}(rv)=0. \qquad \text{... (2.3)}$$

The boundary conditions at the wall require the tangential velocity to be zero and the radial velocity (to be prescribed) v_w (suction or injection).

The boundary conditions for this flow are

at $\quad r = 0,\ \frac{\partial u}{\partial r}=0,\ v = 0,$

at $\quad r=r_w,\ u= 0,\ v=v_w. \qquad$ ... (2.4)

The velocity component v to be taken in the non-dimensional form is

$$v=\frac{V_w f(\eta)}{\eta^{1/2}}, \qquad \text{... (2.5)}$$

where $f(\eta)$ is some function of η and $\eta=\frac{r^2}{r_w^2}$ a non-dimensional variable. Integration of (2.3) yields

$$u=\frac{-2v_w f(\eta)x}{r_w}+u_0(\eta), \qquad \text{... (2.6)}$$

where $u_0(\eta)$ is some arbitrary function of η. Since v is independent of x, and differentiating (2.2) with respect to x, gives

$$\frac{\partial^2 p}{\partial r \partial x} = 0 \text{ or } \frac{\partial^2 p}{\partial \eta \partial x} = 0. \qquad \text{... (2.7)}$$

Using (2.5) and (2.6) into eqs. (2.1) and (2.2) we get[4]

$$f^{iv} + 2f''' + \frac{R}{2}[f' f'' - ff'''] = 0, \qquad \text{... (2.8)}$$

where the prime denotes differentiation with respect to η, and $R = \frac{\rho v_w r_w}{\mu}$ is the wall Reynolds number of the flow.

The boundary conditions (2.4) reduce to

$$f'(1) = 0,\ f(1) = 1,\ f(0) = 0,\ \lim_{n \to 0} \eta^{1/2} f''(\eta) = 0. \qquad \text{... (2.9)}$$

Differential equations of the type (2.8) are solved usually by direct integration which frequently involves more than one integration process, because of the two point nature of the boundary conditions. To confirm the validity of these numerical results they have to be solved by other possible available methods. Besides this, the region of validity of solution can be extended compared with solution obtained using pure numerical methods. Thus, the use of a series solution provides an effective alternative approach. Not only the difficulties associated with the two point boundary value problems are eliminated, but also the terms in the series method are capable of providing results to any desired degree of accuracy with minimum time and storage requirement of computer.

In this paper, we will be concerned with the construction and analysis of a perturbation solution to eq. (2.8) that will be formally valid for small values of Reynolds number R. One of the goals of our analysis is to recast the series in such a manner as to make it valid for values of R as large as possible. Thus the problem reduces in solving the differential equation (2.8) subject to the boundary conditions (2.9).

3. Method of Solution

We seek the solution of (2.8) in power series of R in the form

$$f(\eta) = f_0(\eta) + \sum_{n=1}^{\infty} R^n f_n(\eta). \qquad \text{... (3.1)}$$

Substituting (3.1) into (2.8) and equating the coefficients of like powers of R on both sides, we get

$$\eta f_0^{iv} + 2 f_0''' = 0 \qquad \text{... (3.2)}$$

and

$$\eta f_{n+1}^{iv} + 2f_{n+1}''' = -\frac{1}{2} \sum_{L=0}^{n} f_L' f_{n-L}'' - f_L f_{n-L}''' . \qquad \text{... (3.3)}$$

The relevant boundary conditions take the forms

$$f_0(1) = 1, f_0(0) = 0, f_0'(1) = 0, \lim_{\eta \to 0} \eta^{1/2} f_0''(\eta) = 0$$

$$f_n(1) = f_n(0) = f_n(1) = \lim_{\eta \to 0} \eta^{1/2} f_n''(\eta) = 0, n \geq 1. \qquad \text{... (3.4)}$$

The solutions of the above equation up to $0(R^2)$ are

$$f_0 = \eta(2-\eta),$$

$$f_1 = \frac{1}{9}\eta - \frac{1}{4}\eta^2 + \frac{1}{6}\eta^3 - \frac{1}{36}\eta^4$$

and $$f_2 = \frac{83}{2700}\eta - \frac{19}{270}\eta^2 + \frac{11}{216}\eta^3 - \frac{1}{72}\eta^4 + \frac{1}{360}\eta^5 - \frac{1}{5400}\eta^6. \qquad \text{... (3.5)}$$

3.1. Computer Extended Series

As the series (3.3) is slowly converging it is not reliable to analyse the problem accurately with just few terms. It is essential to get higher approximations. As one proceeds to higher approximations the algebra becomes cumbersome and it is difficult to calculate the terms manually. We propose a systematic series with polynomials coefficients, which is quite useful and efficient in the calculation of higher approximations. The forms of polynomial solutions (3.5) and nature of boundary conditions (3.4) suggest the general form of $f_n(\eta), n \geq 3$ to be of the form

$$f_n(\eta) = \sum_{k=1}^{2n} g_{(n,k)} (1-\eta)^2 \eta^k. \qquad \text{... (3.6)}$$

On substituting (3.6) into (3.3) and equating the coefficients of various powers of η on both sides we get the recurrence relation for generating $g_{(n,k)}$ in the following form :-

$$g_{(n+1, 2n-J)} = 2g_{(n+1, 2n-(J-1))} - g_{(n+1, 2n-(J-2))} +$$

$$\frac{1}{(2n-(J-2))(2n-(J-1))(2n-(J-1))(2n-J)}$$

$$\left[- g_{(n,k)} \sum_{i=1}^{4} P_i (2n - i - J + 2) - \right.$$

$$\left. \sum_{L=1}^{n-1} \left[\sum_{r=-2}^{2} \left[\sum_{k_1 = 2L - J - r}^{2L} g_{(L, k_1)} \, g_{(m, 2n - k_1 - (J + r))} \, P_{7+r} (k_1, 2n - k_1 - (J + r)) \right] \right] \right] \quad \text{... (3.7)}$$

J varies from 0, 1, ... $2n - 1$.

$P_i(KI) = 0$, If $KI > 2n$ ($KI = 2n - i - J + 2$).

$$P_1 (k) = \frac{1}{2} [4k^2 - k^3 - 3k].$$

$$P_2 (k) = \frac{1}{2} k [5k^2 - 9k - 6].$$

$$P_3 (k) = \frac{1}{2} [- 4 k^3 + 12 k + 8].$$

$$P_4 (k) = \frac{1}{2} [k^3 + k^2 - 6k - 8].$$

$$P_5 (k_1, k_2) = \frac{1}{2} [k_1 k_2 (k_2 - 1) - k_2 (k_2 - 1) (k_2 - 2)]$$

$$P_6 (k_1, k_2) = [- k_1 k_2 (k_2 + 1) - k_2 (k_2 - 1) (k_1 + 1) + k_2 (k_2 - 1) (k_2 + 1) + k_2 (k_2 - 1) (k_2 - 2)].$$

$$P_7 (k_1, k_2) = [- k_1 (k_2 + 1) (k_2 + 2) + 4k_2 (k_2 + 1) (k_1 + 1) - k_2 (k_2 + 2) (k_2 + 1) - 4k_2 (k_2 + 1) (k_2 - 1) - k_2 (k_2 - 1) (k_2 - 2) + (k_1 + 2) k_2 (k_2 - 1)].$$

$$P_8 (k_1, k_2) = [- (k_1 + 1) (k_2 + 2) (k_2 + 1) - (k_1 + 2) (k_2 + 1) k_2 + (k_2 + 2) (k_2 + 1) k_2 + (k_2 + 1) k_2 (k_2 - 1)].$$

$$P_9 (k_1, k_2) = \frac{1}{2} [(k_1 + 2) (k_2 + 2) (k_2 + 1) - (k_2 + 2) (k_2 + 1) k_2],$$

$$g(1, 1) = \frac{1}{9}, \; g(1, 2) = - \frac{1}{36} \quad \text{... (3.8)}$$

and

$$g_{(n+1,\,2n+2)} = \frac{1}{X_1}\left[-g_{(n,\,2n)}\,P_4\,(2n) - \sum_{L=1}^{n-1}\left[\sum_{k_1=2L}^{2L} g_{(L,\,k_1)}\,g_{(m,\,2n-k_1)}\;P_9\,(k,\,2n-k_1)\right]\right],$$

$$g_{(n+1,\,2n+1)} = 2\,g_{(n+1,\,2n+2)} + \frac{1}{X_2}\left[-g_{(n,\,2n)}\,P_3\,(2n) - g_{(n,\,k)}\,P_4\,(2n-1) - \sum_{L+1}^{n-1}\right.$$

$$\left.\left[\sum_{k_1=2L}^{2L} g_{(L,\,k_1)}\,g_{(m,\,2n-k_1)}\,P_8\,(k_1,\,2n-k_1) + \sum_{k_1=2L-1}^{2L} g_{(L,\,k_1)}\,g_{(m,\,2n-1-k_1)}\,P_9\,(k_1,\,2n-1-k_1)\right]\right],$$

... (3.9)

where

$$X_1 = (2n+4)\,(2n+3)\,(2n+3)\,(2n+2),$$

$$X_2 = (2n+3)\,(2n+2)\,(2n+2)\,(2n+1).$$

The recurrence relation is implemented using one of the arithmetic languages (FORTRAN) in getting universal coefficients $g_{(n,\,k)}$.

The skin-friction at the wall is represented by the series

$$f''(1) = -2 + \sum_{n=1}^{\infty} R^n \sum_{k=1}^{2n} 2\,g_{(n,\,k)} \qquad \text{... (3.10)}$$

The expression for axial velocity profiles is given by

$$f'(\eta)\,2 - 2\eta + \sum_{n=1}^{\infty} R^n \sum_{k=1}^{2n} g_{(n,\,k)}\left[k\,\eta^{k-1} - 2(k+1)\eta^k + (k+2)\,\eta^{k+1}\right] \qquad \text{... (3.11)}$$

3.2. Analysis and Improvement of Series

Euler Transformation —

The expression for skin friction is given by

$$f''(1) = -2 + \sum_{n=1}^{\infty} E_n R^n, \qquad \text{... (3.12)}$$

where

$$E_n = \sum_{k=1}^{\infty} 2\,g_{(n,\,k)}$$

and for axial velocity profiles

$$f'(\eta) = 2 - 2\eta + \sum_{n=1}^{\infty} R^n \sum_{k=1}^{2n} g_{(n,k)} \left[k\,\eta^{k-1} - 2(k+1)\,\eta^k + (k+2)\,\eta^{k+1} \right] \ldots \quad (3.13)$$

For negative Reynolds numbers the coefficients of the series given by eqs. (3.10) (Table I)

TABLE I

Coefficients of series (3.12) representing skin friction For negative Reynolds numbers

n	c_n	n	c_n
0	-2.0000000000000	18	-1.6719067349978E-010
1	1.6666666666667E-001	19	4.9585329048721E-011
2	-4.8148148148148E-002	20	-1.4705992896848E-011
3	1.4065255731922E-002	21	4.3614962577030E-012
4	-4.1317117381932E-003	22	-1.2935304497565E-012
5	1.2192658878616E-003	23	3.8363463490139E-013
6	-3.6117061776392E-004	24	-1.1377817439368E-013
7	1.0707901747340E-004	25	3.3744275908995E-014
8	-3.1754932909705E-005	26	-1.0007861021592E-014
9	9.4176879024574E-006	27	2.9681265793823E-015
10	-2.7930807784341E-006	28	-8.8028554475612E-016
11	8.2836960359552E-007	29	2.6107466092900E-016
12	-2.4567738352673E-007	30	-7.7429396614794E-017
13	7.2862873932878E-008	31	2,2963972983045E-017
14	-2.1609635470525E-008	32	-6.8106439959685E-018
15	6.4089761999825E-009	33	2.0198975009276E-018
16	-1.9007707613943E-009	34	-5.9906022347795-019
17	5.6372967253077E-010	35	1.7766899121794E-019

and (3.11) are deceasing in magnitude and have alternate sign pattern. The nearest singularity, lying on the negative axis has, no direct physical significance. In this case, the simplest device to use is an Euler transformation based on the estimate of ε_0, the radius of convergence of the series (3.12). With this transformation, the singularity is mapped to infinity. The transformation envisages using a new variable ε^* such that

$$\varepsilon^* = \frac{R}{R + \varepsilon_0}$$

or

$$R = \frac{\varepsilon_0 \, \varepsilon^*}{1 - \varepsilon^*}.$$

The series (3.12) is rearest into new form

$$f''(1) = \sum_{n=0}^{\infty} b_n \, \varepsilon^{*n}. \qquad \text{... (3.14)}$$

where the coefficients b_n are given by

$$b_0 = -2$$

$$b_n = \sum_{j=1}^{n} \frac{(n-1)\,!}{(n-j)\,!\,(j-1)\,!} a_j \, \varepsilon_0^j. \qquad \text{... (3.15)}$$

The first five coefficients can be written as

$$f''(1) = a_0 + (a_1\,\varepsilon_0)\varepsilon^* + (a_1\varepsilon_0 + a_2\varepsilon_0)\,\varepsilon^{*2} + (a_1\varepsilon_0 + 2a_2\,\varepsilon_0^2 + a_3\varepsilon_0^3)\,\varepsilon^{*3} +$$

$$(a_1\varepsilon_0 + 3a_2\varepsilon_0^2 + 3a_3\varepsilon_0^3 + a_4\varepsilon_0^4)\,\varepsilon^{*4} + \text{............}$$

The new series (3.12) can be used to approximate the solution up to $R = -300$. The similar analysis is carried for recasting the series (3 11) into new Eularised form.

4. Results and Discussion

To analyse the accuracy of the series solution the results obtained are compared with the numerical solution[4]. The values of skin friction (shear stress at the wall) and non-dimensional centre line velocity are calculated. For large injection through the wall the coefficients of the series (3.10) and (3.11) (which represents skin-friction and velocity profiles) are alternating in sign. A plot of $|c_n|^{1/n}$ versus $1/n$ (Domb-Sykes plot[8]) is shown in Fig 2. Domb-Sykes plot shows that the singularity restricting the convergence of the series (3.10) is a simple pole. Extraction using rational approximation (William *et al.*[9]) yields the radius of convergence of the series to be $\varepsilon_0 = 3.33$. The utility of the series is improved using Euler transformation. The series (3.14) is summed using Pade' approximants. The application of the Euler transformation and Pade' approximants improves the region of validity of the series from $-241 \le R \le -24$ (numerical[4]) to $-300 \le R \le 1$ (Table II). Similar type of analysis is performed for the axial velocity profiles also. In the case of injection through the wall the flow is stabilized and the transition to the turbulence is delayed.

TABLE II

Comparison of shear stress ($f''(1)$) values at the wall, obtained by series solutions with those of available numerical results

R	$f''(1)$ series	$f''(1)$ numerical
1	–1.7652	-
2	–1.1996	-
2.55	0	-
–1	–2.1293	-
–20	–2.5000	-
–24.19906	–2.5145	–2.4571
–33.72519	–2.5345	–2.4637
–44.66713	–2.5478	–2.4667
–55.6083	–2.5561	–2.4680
–75	–2.5650	-
–80.36234	–2.5668	–2.4690
–91.64563	–2.5699	–2.4690
–102.9344	–2.5722	–2.4691
–120	–2.5750	-
–140	–2.5773	-
–151.3834	–2.5782	–2.4689
–170	–2.5798	-
–190	–2.5820	-
–210	–2.5815	-
–230	–2.5828	-
–241.9763	–2.5829	–2.4685
–250	–2.5828	-
–275	–2.5835	-
–300	–2.5840	-

In the case of suction through two walls, the numerical solutions show the non-existence of solution in the range $2.3 < R < 9.1$. At $R = 2.3$ the skin friction becomes zero and for $R > 2.3$ there is change in its sign. Eckert[10] *et al.* could extend the range to $2 < R < 10$. Berman[1] had results in the range $2.4 < R < 9$. From our long series analysis the separation point located is at $R = 2.55$. For $R > 2.55$ the skin friction changes its sign. We are able to find solution up to the point where $f''(1)$ becomes zero. Beyond this point a power series expansion fails.

Fig. 3 shows velocity profiles for large injection and for small suction Reynolds number. Centre line velocity is around 1.6 for large injection Reynolds number (- 150) whereas for small suction Reynolds number ($R = 2.55$) it attains the velocity around 3.3.

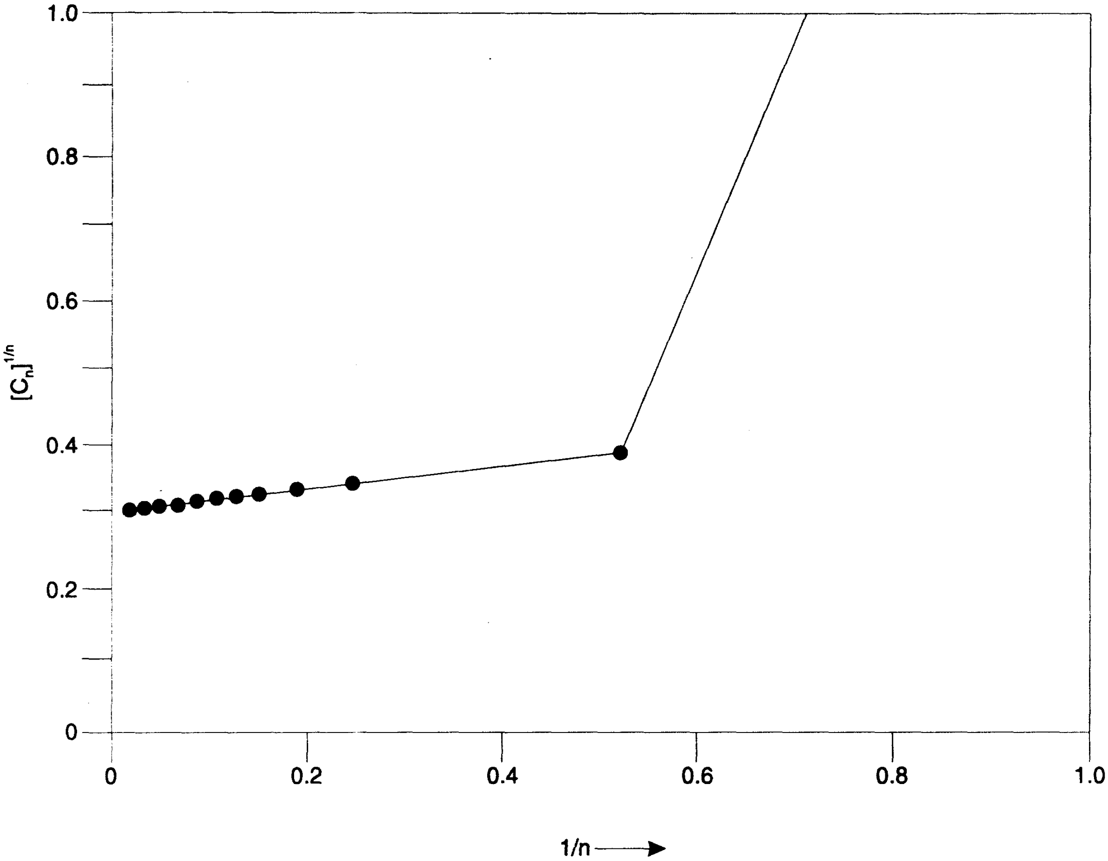

FIG. 2. Domb-Sykes plot (For skin-friction and axial velocity profiles).

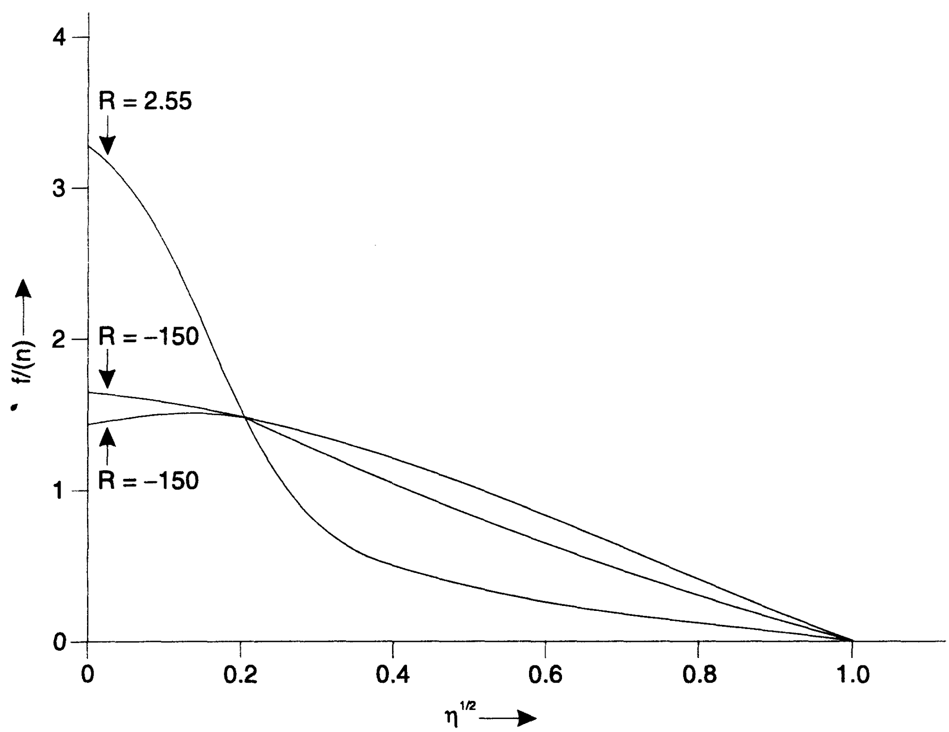

FIG. 3. Axial velocity $f'(\eta)$ against nondimensional radial distance $\eta^{1/2}$ for wall suction and injection

CONCLUSION

The method proposed here is quite flexible and can be efficiently implemented on a computer compared to any other pure numerical methods. Once the universal coefficients are generated the rest of the analysis can be done efficiently. The analytic structure of solution can also be obtained. Also, any derived quantity can be obtained easily whereas in the case of numerical schemes it requires elaborate additional numerical schemes. The method requires less time and storage.

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