

# A CHARACTERIZATION OF CLOSED RANGE OPERATORS\*

S. H. KULKARNI AND M. T. NAIR

*Department of Mathematics, Indian Institute of Technology Madras,  
Chennai 600 036, India*

*E-Mail Addresses : shk@acer.iitm.ernet.in, mtnair@acer.iitm.ernet.in*

*(Received 7 August 1998; Accepted 25 May 1999)*

We prove that if  $T: X \rightarrow Y$  is a bounded linear operator between Hilbert spaces  $X$  and  $Y$ , then the range of  $T$  is closed in  $Y$  if and only if  $0$  is not an accumulation point of the spectrum of  $T^*T$ . We supply three proofs of this result as applications of different concepts in basic operator theory.

**Key Words :** Closed Range Operators; Accumulation Point; Generalized Inverse; Ill-posed, Spectral Theorem; Spectral Projection; Tikhonov Regularization

## 1 INTRODUCTION

Let  $X$  and  $Y$  be Hilbert spaces and  $T: X \rightarrow Y$  be a bounded linear operator. It is well known that if  $T: X \rightarrow Y$  is a compact operator, then the range  $R(T)$  of  $T$  is closed in  $Y$  if and only if  $0$  is not an accumulation point of the spectrum  $\sigma(T^*T)$  of  $T^*T$ .

In this note, we prove the above result for a general bounded linear operator. This gives a characterization of bounded operators that have closed ranges.

It should be mentioned that the knowledge whether the range of an operator  $T$  is closed or not is particularly important in the context of solving the operator equation

$$Tx = y.$$

In fact, the above equation is ill-posed if the range of  $T$  is not closed. For, in that case the generalized inverse  $T^\dagger: R(T) + R(T)^\perp \rightarrow X$  which maps  $y \in R(T) + R(T)^\perp$  to  $T^\dagger y$ , the least-square solution of minimal norm, is discontinuous (c.f. Groetsch<sup>3&4</sup>).

We give three proofs of the above characterization of the closedness of  $R(T)$ , as applications of different concepts in basic operator theory. To start with we show (see Theorem 2.1) that

$R(T)$  is closed  $\Leftrightarrow 0$  is not an accumulation point of  $\sigma\left(T^*T|_{N(T)^\perp}\right)$ , where  $N(T)^\perp$  denotes the orthogonal complement of the null space  $N(T)$  of  $T$ . Since

\*The work of M.T. Nair is partially supported by National Board for Higher Mathematics (Department of Atomic Energy) Government of India.

$$\sigma(T^*T) \subseteq \{0\} \cup \sigma\left(T^*T|_{N(T)^\perp}\right)$$

it follows immediately that if  $R(T)$  is closed, then 0 is not an accumulation point of  $\sigma(T^*T)$ . Thus, in essence, the three different proofs are, in fact, the proofs of the reverse implication, namely, if 0 is not an accumulation point of  $\sigma(T^*T)$ , then  $R(T)$  is closed. For this, we shall make use of certain results which are important in their own right as well.

The first proof is by making use of the fact that an isolated spectral value of a self adjoint operator is an eigenvalue. Although this is a consequence of the spectral theorem (c.f. Limaye<sup>7</sup>, Corollary 32.9) we shall give an alternate proof using the concept of spectral projection, by assuming that the scalar field is the set of complex numbers.

The second proof is motivated by the procedure of Tikhonov regularization of an ill-posed operator equation. This proof essentially makes use of the Closed Graph Theorem, the Uniform Boundedness Principle, and the facts that (i) the spectral radius and norm of a self-adjoint operator are the same and (ii) the nonzero spectral values of  $T^*T$  and  $TT^*$  are the same.

In the third proof, we use a special form of the spectral theorem, namely, the result that every self-adjoint operator can be represented as a multiplication operator on an  $L^2$ -space.

It is to be mentioned that a particular case of the characterization theorem, namely if  $T$  is injective, then  $T^{-1}:R(T) \rightarrow X$  is continuous if and only if 0 is not an accumulation point of  $\sigma(T^*T)$ , has been proved by Hegland<sup>6</sup> using the spectral representation of  $T^*T$ .

## 2. THE MAIN RESULT

In the following,  $X$  and  $Y$  denote Hilbert spaces over the field of real or complex numbers. For  $S \subseteq X$ , the closure of  $S$  is denoted  $\bar{S}$ , and the set  $\{x \in X : \langle x, u \rangle = 0, \forall u \in S\}$  is denoted by  $S^\perp$ . For a bounded linear operator  $A : X \rightarrow Y$ , we denote the null space of  $A$  by  $N(A)$  and the range of  $A$  by  $R(A)$ . In case  $Y = X$ , then the spectrum of  $A$  is denoted by  $\sigma(A)$ .

For a bounded linear operator  $A : X \rightarrow Y$ , the following results are well known:-

- $A^*A : X \rightarrow X$  is a positive self-adjoint operator.
- $R(A)$  is closed if and only if  $R(A^*A)$  is closed.

$$\overline{R(A^*)} = N(A)^\perp = N(A^*A)^\perp = \overline{R(A^*A)}.$$

- $N(A)$  and  $N(A)^\perp$  are invariant under  $A^*A$  and

$$\sigma(A^*A) = \sigma(A^*A|_{N(A)}) \cup \sigma\left(A^*A|_{N(A)^\perp}\right) \subseteq \{0\} \cup [0, \|A\|^2].$$

First we give a characterization of the closedness of  $R(T)$  in terms of the behaviour of the spectrum of  $T^*T|_{N(T)^\perp}$  in the neighbourhood of 0.

**Theorem 2.1** — *Let  $T : X \rightarrow Y$  be a nonzero bounded linear operator. The  $R(T)$  is closed in  $Y$  if and only if there exists  $\gamma > 0$  such that*

$$\sigma\left(T^*T|_{N(T)^\perp}\right) \subseteq [\gamma, \|T\|^2],$$

and in that case

$$\sigma(T^*T) \subseteq \{0\} \cup [\gamma, \|T\|^2].$$

PROOF : Recall that  $N(T)^\perp = N(T^*T)^\perp = \overline{R(T^*T)}$  and it is invariant under  $T^*T$ . Consider the operator

$$A := T^*T|_{N(T)^\perp}: N(T)^\perp \rightarrow N(T)^\perp.$$

Clearly,  $A$  is injective and its range  $R(T^*T)$  is dense in  $N(T)^\perp$ . Now,

- $R(T)$  is closed  $\Leftrightarrow R(T^*T)$  closed
- $\Leftrightarrow A$  is bijective
- $\Leftrightarrow 0 \notin \sigma(A)$
- $\Leftrightarrow \exists \gamma > 0$  such that  $\sigma(A) \subseteq [\gamma, \|T\|^2]$ ,

and in that case, we have

$$\sigma(T^*T) = \sigma(T^*T|_{N(T)}) \cup \sigma\left(T^*T|_{N(T)^\perp}\right) \subseteq \{0\} \cup [\gamma, \|T\|^2]. \quad \square$$

The proof of the following result makes use of the concept of spectral projection associated with an isolated spectral value. We recall that if  $\lambda$  is an isolated spectral value of a bounded linear operator  $A : X \rightarrow X$ , where  $X$  is a complex Hilbert space, then the spectral projection  $P$  associated with  $\lambda$  is given by

$$P = \frac{-1}{2\pi i} \int_{\Gamma} (A - zI)^{-1} dz,$$

where  $\Gamma$  is a simple closed contour not passing through any of the spectral values such that  $\lambda$  is the only spectral value lying inside  $\Gamma$  (c.f. Limaye<sup>8</sup>). In fact, the above operator  $P$  is a non-zero projection, since  $\lambda$  lies inside  $\Gamma$ .

*Proposition 2.2* — Let  $X$  be a complex Hilbert space and  $A : X \rightarrow X$ , be bounded self adjoint operator on  $X$ . Then every isolated spectral value of  $A$  is an eigenvalue of  $A$ .

PROOF : Let  $\lambda$  be an isolated spectral value of  $A$ , and let  $r > 0$  be such that

$$\sigma(A) \cap \{z \in \mathbb{C} : |z - \lambda| < r\} = \{\lambda\}.$$

Let  $\Gamma = \{z \in \mathbb{C} : |z - \lambda| = r/2\}$  and  $P$  be the spectral projection associated with  $\lambda$ , i.e.,

$$P = \frac{-1}{2\pi i} \int_{\Gamma} (A - zI)^{-1} dz.$$

Then, using Cauchy's theorem for operator valued analytic functions, we have

$$\begin{aligned} (A - \lambda I) P &= \frac{-1}{2\pi i} \int_{\Gamma} (A - \lambda I) (A - zI)^{-1} dz \\ &= \frac{-1}{2\pi i} \int_{\Gamma} [I - (\lambda - z) (A - zI)^{-1}] dz \\ &= \frac{1}{2\pi i} \int_{\Gamma} (\lambda - z) (A - zI)^{-1} dz. \end{aligned}$$

Since  $A$  is a self adjoint operator, we have

$$\| (A - zI)^{-1} \| = \frac{1}{\text{dist}(z, \sigma(A))} = \frac{1}{|\lambda - z|} \text{ for } z \in \Gamma$$

Hence, using the notation  $l(\Gamma)$  for the length of  $\Gamma$ , we obtain

$$\begin{aligned} \| (A - \lambda I) P \| &\leq \frac{l(\Gamma)}{2\pi} \max_{z \in \Gamma} |\lambda - z| \| (A - zI)^{-1} \| \\ &\leq \frac{l(\Gamma)}{2\pi} \max_{z \in \Gamma} \frac{|\lambda - z|}{\text{dist}(z, \sigma(A))} \\ &= \frac{r}{2}. \end{aligned}$$

Letting  $r \rightarrow 0$ , it follows that  $AP = \lambda P$ , so that  $\lambda$  is an eigenvalue of  $T$  with  $R(P)$  as the corresponding eigenspace. □

It is to be mentioned that, although the above result is stated for a complex Hilbert space, it still holds for a real Hilbert space (c.f. Limaye<sup>7</sup>). Moreover, the above proof holds for a normal operator on a complex Hilbert space as well, since the only place in the proof where the self-adjointness of the operator used is for the equality

$$\| (A - zI)^{-1} \| = \frac{1}{\text{dist}(z, \sigma(A))}, \quad z \notin \sigma(A),$$

which is true for a normal operator  $A$ .

Let  $T: X \rightarrow Y$  be a bounded linear operator. Let

$$D = R(T) + R(T)^\perp, \quad R_\alpha = (T^*T + \alpha I)^{-1} T^*, \quad \alpha > 0$$

and  $T^\dagger : R(T) + R(T)^\perp \rightarrow X$  be the generalized inverse of  $T$ .

*Proposition 2.3* — The following are equivalent:

- (i)  $T^\dagger : D \rightarrow X$  is a bounded linear operator;
- (ii)  $R(T)$  is a closed subspace of  $Y$ ; and
- (iii)  $\{R_\alpha : \alpha > 0\}$  is uniformly bounded.

**PROOF :** The equivalence (i) and (ii) is a well-known result (c.f. Groetsch<sup>3</sup>). Its proof is along the following line. First show that  $T^\dagger$  is a closed linear operator, and then use the Closed Graph Theorem.

Now we prove the equivalence of (ii) and (iii). For this, we make use of the convergence

$$R_\alpha y \rightarrow T^\dagger y$$

for every  $y \in D$ . A proof of this fact is available in Schock<sup>9</sup>. However, for the sake of completeness we give its details.

For  $y \in D$ , let

$$\hat{x} = T^\dagger y \text{ and } x_\alpha = R_\alpha y.$$

Then we have

$$(T^*T + \alpha I)\hat{x} = T^*y + \alpha\hat{x} \text{ and } (T^*T + \alpha I)x_\alpha = T^*y,$$

so that

$$\hat{x} - x_\alpha = \alpha(T^*T + \alpha I)^{-1} \hat{x}.$$

Now, let  $A_\alpha = \alpha(T^*T + \alpha I)^{-1}$ ,  $\alpha > 0$ . Then for every  $x \in X$ ,

$$\|A_\alpha T^*Tx\| \leq \alpha \|(T^*T + \alpha I)^{-1} T^*T\| \|x\|.$$

But using spectral properties of the self adjoint operator  $T^*T$ , we have

$$\|(T^*T + \alpha I)^{-1} T^*T\| \leq \sup \{ \lambda / (\lambda + \alpha) : \lambda \in \sigma(T^*T) \} \leq 1.$$

Thus it follows that  $\|A_\alpha u\| \rightarrow 0$  as  $\alpha \rightarrow 0$  for every  $u \in R(T^*T)$ . Since  $R(T^*T)$  is dense in  $N(T)^\perp$ ,  $\hat{x} \in N(T)^\perp$  and  $\|A_\alpha\| \leq 1$  for every  $\alpha > 0$ , it follows that

$$\|\hat{x} - x_\alpha\| = \|A_\alpha \hat{x}\| \rightarrow 0 \text{ as } \alpha \rightarrow 0.$$

Thus  $R_\alpha y \rightarrow T^\dagger y$  for every  $y \in D$ .

Now assume that  $\{R_\alpha\}_{\alpha > 0}$  is uniformly bounded. Then using the fact that  $D$  is dense in  $Y$  and the convergence  $R_\alpha y \rightarrow T^\dagger y$  for every  $y \in D$ , imply that  $(R_\alpha y)$  converges for every  $y \in Y$  and

the limiting operator  $R_0 : Y \rightarrow X$  defined by

$$R_0 y := \lim_{\alpha \rightarrow 0} R_\alpha y, y \in Y,$$

is a bounded linear operator. But  $R_0 y = T^\dagger y$  for every  $y \in D$ . Hence  $T^\dagger$ , the restriction of a bounded linear operator, is a bounded linear operator. Therefore, by the equivalence of (i) and (ii),  $R(T)$  is closed in  $Y$ .

Conversely, suppose that  $R(T)$  is closed in  $Y$ . Then  $D = Y$  so that  $(R_\alpha y)$  converges for every  $y \in Y$ . Hence, by the Uniform Boundedness Principle,  $\{R_\alpha\}_{\alpha > 0}$  is uniformly bounded. □

For a multiplication operator on an  $L^2$ -space, the required result is obtained from the next Proposition.

*Proposition 2.4* — Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space with  $\mu$  being a positive measure,  $g \in L^\infty(\Omega, \mathcal{A}, \mu)$  and

$$A : L^2(\Omega, \mathcal{A}, \mu) \rightarrow L^2(\Omega, \mathcal{A}, \mu) \text{ be defined by } Af = gf, f \in L^2(\Omega, \mathcal{A}, \mu).$$

Let

$$E := \{x \in \Omega : g(x) = 0\} \text{ and } R := \{f \in L^2(\Omega, \mathcal{A}, \mu) : f(x) = 0, \forall x \in E\}.$$

Then  $\overline{R(A)} = R$ . Also, the following statements are equivalent.

- (i)  $R(A)$  is a closed subspace of  $L^2(\Omega, \mathcal{A}, \mu)$ .
- (ii)  $R(A) = R$ .
- (iii)  $\exists \gamma > 0$  such that  $|g| \geq \gamma$  a.e. on  $\Omega \setminus E$ .
- (iv)  $\exists \gamma > 0$  such that  $\sigma(A) \cap \{z \in \mathbb{C} : |z| < \gamma\} \subseteq \{0\}$ .

PROOF : First we note that  $R(A) \subseteq R$  and  $R$  is a closed subset of  $L^2(\Omega, \mathcal{A}, \mu)$ . Consider the sets

$$F := \{x \in \Omega : g(x) \neq 0\} \text{ and } F_n := \{x \in \Omega : |g(x)| > 1/n\}, n = 1, 2, \dots$$

Define

$$h = (1/g)\varphi \text{ and } h_n = (1/g)\varphi_n,$$

where  $\varphi$  and  $\varphi_n$  are the characteristic functions of  $F$  and  $F_n$  respectively. Then  $h$  and  $h_n$  are measurable.

Now let  $f \in R$ . Since  $|h_n| \leq n$ , we have  $h_n f \in L^2(\Omega, \mathcal{A}, \mu)$ . Also

$$A(h_n f) = \varphi_n f \rightarrow \varphi f = f$$

pointwise. Hence by the Dominated Convergence Theorem,  $\varphi_n f \rightarrow f$  in  $L^2(\Omega, \mathcal{A}, \mu)$  so that  $f \in \overline{R(A)}$ . Thus  $\overline{R(A)} = R$ . This also proves the equivalence of (i) and (ii). The equivalence of (iii)

and (iv) follows from a well-known fact (c.f. Limaye<sup>7</sup>, 27.4 (b)) that  $\sigma(A)$  is the essential range of  $g$ . Thus it is enough to prove the equivalence of (ii) and (iii).

Suppose (iii) holds. Then it follows that  $|h| \leq 1/\gamma$  a.e. on  $\Omega$ . Hence for  $f \in R$ , we have

$$fh \in L^2(\Omega, \mathcal{A}, \mu) \text{ and } A(fh) = ghf = f,$$

so that  $f \in R(A)$ . Hence, (iii) implies (ii).

Now assume that (ii) holds, and define

$$B : L^2(\Omega, \mathcal{A}, \mu) \rightarrow L^2(\Omega, \mathcal{A}, \mu) \text{ by } Bf = hf, f \in L^2(\Omega, \mathcal{A}, \mu).$$

Then a straightforward application of the Closed Graph Theorem shows that  $B$  is bounded. (Alternatively, one can observe that  $B = A^\dagger$  and use Proposition 2.3.) Also,

$$\|B\| = \|h\|_\infty = \|(1/g)\varphi\|$$

and therefore, (iii) follows by taking  $\gamma = 1/\|h\|_\infty$ . □

Now we deduce three proofs of the main result :

**Theorem 2.5** — *Let  $X$  and  $Y$  be complex Hilbert spaces and  $T : X \rightarrow Y$  be a bounded linear operator. Then*

$$R(T) \text{ is closed in } Y \Leftrightarrow \exists \gamma > 0 \text{ such that } \sigma(T^*T) \subseteq \{0\} \cup [\gamma, \|T\|^2].$$

PROOF : We have already observed in Theorem 2.1 that if  $R(T)$  closed in  $Y$ , then there exists  $\gamma > 0$  such that  $\sigma(T^*T) \subseteq \{0\} \cup [\gamma, \|T\|^2]$ .

Conversely, assume that there exists  $\gamma > 0$  such that

$$\sigma(T^*T) \subseteq \{0\} \cup [\gamma, \|T\|^2].$$

To show that  $R(T)$  is closed in  $Y$ . We give three proofs for this.

PROOF (1) : By Theorem 2.1, its enough to prove that  $0 \notin \sigma(T^*T|_{N(T)^\perp})$ .

If  $0 \in \sigma(T^*T|_{N(T)^\perp})$ , then it follows that 0 is an isolated spectral value of the self adjoint operator  $T^*T|_{N(T)^\perp} : N(T)^\perp \rightarrow N(T)^\perp$ . Now, by Proposition 2.2, 0 is an eigen value of  $T^*T|_{N(T)^\perp}$  which is not possible.

PROOF (2) : Recall that  $R_\alpha = (T^*T + \alpha I)^{-1} T^*$ ,  $\alpha > 0$ . Then it follows that

$$R_\alpha^* R_\alpha = T(T^*T + \alpha I)^{-2} T^* = (TT^* + \alpha I)^{-2} TT^*,$$

so that

$$\|R_\alpha\|^2 = \|R_\alpha^* R_\alpha\| = \sup \{ \lambda(\lambda + \alpha)^{-2} : \lambda \in \sigma(TT^*) \}, \forall \alpha > 0.$$

Now by observing that

$$\sigma(T^*T) \cup \{0\} = \sigma(TT^*) \cup \{0\}$$

and

$$\frac{\lambda}{(\lambda + \alpha)^2} \leq \frac{1}{\lambda} \leq \frac{1}{\gamma}, \quad 0 \neq \lambda \in \sigma(T^*T), \quad \forall \alpha > 0,$$

it follows that

$$\|R_\alpha\|^2 \leq \frac{1}{\gamma}, \quad \forall \alpha > 0.$$

Hence, by Proposition 2.3,  $R(T)$  is closed in  $Y$ .

PROOF (3) : Recall from spectral theory (c.f. [2]) that if  $X$  is a complex Hilbert space and  $A : X \rightarrow X$  is a self adjoint operator on a Hilbert space  $X$ , then there exists a measure space  $(\Omega, \mathcal{A}, \mu)$  and a subjective isometry

$$J : X \rightarrow L^2(\Omega, \mathcal{A}, \mu)$$

such that the self adjoint operator  $\tilde{A} := JAJ^{-1} : L^2(\Omega, \mathcal{A}, \mu) \rightarrow L^2(\Omega, \mathcal{A}, \mu)$  is a multiplication operator, i.e.,

$$\tilde{A}f = g \cdot f, \quad f \in L^2(\Omega, \mathcal{A}, \mu),$$

for some real valued  $g \in L^\infty(\Omega, \mathcal{A}, \mu)$ . (See also [5] for a very interesting account of this version of the Spectral Theorem). Thus,  $R(A)$  is closed if and only if  $R(JAJ^{-1})$  is closed, so that the proof follows from Proposition 2.4 by taking  $A = T^*T$ .  $\square$

*Remarks* : Although the main result, Theorem 2.5, is stated for complex Hilbert spaces  $X$  and  $Y$ , it is true for real Hilbert spaces as well. Clearly, the proof of the first part of the theorem relies on Theorem 2.1 which is proved for general Hilbert spaces. To see the validity of the second part of the theorem, we observe the following :-

(i) As already mentioned earlier, Proposition 2.2 is true for a real Hilbert space as well (cf. Limaye<sup>7</sup>, Corollary 32.9), as a consequence of the spectral theorem for a bounded self adjoint operator. Thus, *Proof (1)* holds for real Hilbert spaces as well.

(ii) *Proof (2)* essentially makes use of Proposition 2.4 and a result from [2] on representation of a self adjoint operator as a multiplication operator on an  $L^2$ -space. In Proposition 2.4, the scalar field can be real or complex, whereas the stated result from [2] to be valid for a real Hilbert space, some modifications are required. These modifications are available in Aggarwal and Kulkarni<sup>1</sup>.

## REFERENCES

1. S. Aggarwal and S. H. Kulkarni, *Math. Today* **161** (1997), 65-76.
2. N. Dunford and J. T. Schwartz, *Linear Operators Part II Interscience*, New York, 1963.



3. C. W. Groetsch, *Generalized Inverse of Linear Operators : Representation and Application*, Marcel Dekker, New York. 1977.
4. C. W. Groetsch, *The Theory of Tikhonov Regularization for Fredholm Integral Equations of the First Kind*, Pitman, Boston, London, Melbourne, 1984.
5. P. R. Halmos, *Amer. Math. Month.*, **70** (1963), 241-247.
6. M. Hegland, *Appl. Anal.* **59** (1995), 207-223.
7. B. V. Limaye, *Functional Analysis*, Wiley-Eastern Ltd., 1981. (Revised Printing 1984).
8. B. V. Limaye, *Spectral Perturbation and Approximation with Numerical experiments*, *Proc. Cent. Math. Anal.* Australian National University, Vol. 13, 1986.
9. E. Schock., *Studia Mathematica*, **T.LXXXI** (1985) 1-11.