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STABILITY OF A STRATIFIED FLUID THROUGH POROUS MEDIA THE PRESENCE OF SUSPENDED PARTICLES, ROTATION AND VERTICAL OSCILLATIONS

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The stability of a stratified, incompressible fluid contained between two horizontal planes through porous media in the presence of suspended particles, rotation and vertical oscillations is considered. In the absence of porous media, it is found that, rotation and vertical oscillations have a stabilizing effect on the stability of a stratified fluid while suspended particles may be stabilizing or destabilizing, depending on the parameter defining the direction of mass concentration of suspended particles. It is also found that the porosity has a destabilizing effect in the presence of rotation and vertical oscillations and in the absence of suspended particles.

Key Words : Stability; Stratified Fluid; Porous Media; Suspended Particles; Oscillation - Rotation and Vertical; Perturbation Equations

1. INTRODUCTION

The stability of an incompressible heavy fluid of variable density (i.e., heterogeneous fluid) was investigated by Lord Rayleigh[1].

He showed that, a stratified liquid, the density of which increases exponentially in the vertical direction, gravitational acceleration acting vertically downward, is unstable against disturbances of all wavelengths. Chandrasekhar[2] showed that the rotation of a stratified liquid about the vertical direction stabilizes the system against disturbances of sufficiently large horizontal wavelengths. Wesson[3] showed that the system becomes stable against all disturbances except those having small vertical wavelengths in the presence of vertical oscillations. Chakraborty and Bandyopadhyay[4] found that, in general, rotation, combined with vertical oscillations, has a stabilizing influence, and, in particular, the short vertical wavelength disturbances, which are unstable in the presence of vertical oscillations alone, can be stabilized under suitable conditions[4].

In all the above studies, the medium has been considered to be nonporous. The flow of a fluid through porous medium has gained considerable interest in recent years specially among geophysical fluid dynamicists and petroleum engineers with reference to oil recovery from the earth. The effect of suspended particles is widely considered as it has been observed that interstellar media contain grains which are small particles formed in the outer atmosphere of stars and ejected into the medium.

Alfven *et al.*[5] have emphasized the inclusion of dust in the analysis of the problem of star formation. Recently, Kirti and Seema[6] Studied Rayleigh-Taylor instability of an infinite, incompressible, homogeneous, conducting fluid in porous medium in the presence of uniform rotation and suspended particles[6].

The aim of this paper is to study the effect of suspended particles, rotation and vertical oscillations on the stability of a stratified fluid in porous medium.

2. Formulation of the Problem and Perturbation Equations

Consider a composite homogeneous medium consisting of fluid and particles. The medium has infinite extension along x and y directions and is bounded by two rigid planes at $z = 0$ and $z = d$. The medium is assumed to be incompressible. The fluid and the planes rotate with uniform angular velocity Ω about z-axis and have vertical oscillations with acceleration $a\omega_s^2 \cos \omega_s t$ in a frame of reference at rest, where a and ω_s are constants. If the particles are assumed to be uniform size, spherical shape and have small relative velocities between the two phases, then an extra body force per unit volume $KN(v\text{-}u)$ is added to the momentum transfer equation for the fluid as the effect of suspended particles on the fluid, where $K = 6\pi\rho_0 \nu\eta$; the quantities η, ν, u and v denote the particle radius, kinematic viscosity of the clean fluid, the velocity of the fluid and the velocity of the particles, respectively (cf. Scanlon *et al.*[7]. The density of the fluid and number density of the particles are represented by ρ_0 and N. The interparticle reactions are neglected as the distance between two particles is assumed to be large in comparison to their diameters.

When the fluid slowly filters through the pores of macroscopically homogeneous and isotropic porous medium of permeability k_1, the gross effect is represented by Darcy's law according to which, the usual viscous term in the equations of fluid motion is replaced by the resistance term $\left(\frac{\mu}{k_1}\right)\boldsymbol{u}$ where $\mu(=\rho_0\nu)$ is the medium viscosity of the fluid.

We use a coordinate system, with the z-axis in the vertical direction, rotating with angular velocity Ω about the z-axis and oscillating vertically with acceleration $a\omega_s^2 \cos \omega_s t$. In this frame of reference the fluid, situated between the planes $z = 0$ and $z = d$, is at rest before the disturbances and is subjected to gravitational acceleration $g(t) = -g_0 - a\omega_s \cos \omega_s t$ along the z-axis[8]. The fluid density prior to the disturbances is given by $\rho_0 = f \exp(z/\lambda)$.

Therefore, the linearized perturbation equations of the fluid particle medium are

$$\rho_0 \frac{\partial u}{\partial t} = -\nabla p + 2\rho_0 (u x \Omega) + KN(v-u) - \frac{\rho_0 \nu}{k_1} u + g(t)\,\rho\, \boldsymbol{e}_z, \qquad \dots (1)$$

$$\left(\tau_1 \frac{\partial}{\partial t} + 1\right) v = u, \qquad \dots (2)$$

$$\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho_0 = 0 \qquad \dots (3)$$

and

$$\nabla \cdot u = 0. \qquad \dots (4)$$

Where $u(u_x, u_y, u_z)$, p, ρ denote the perturbations in the velocity, the pressure, and the density ρ_0, respectively; $\tau_1(=m/K)$ represents relaxation time for the suspended particles. The stability of the system is studied against small disturbances which are given by

$$q = \hat{q}(z, t) \exp \cdot [i(k_x x + k_y y)], \quad \dots (5)$$

where $\hat{q}(z, t)$ is some function of z and t, k_x and k_y are horizontal wave numbers, $k_x^2 + k_y^2 = k^2$.

From (4) we have

$$ik u_x' + \frac{\partial u_z}{\partial z} = 0, \quad \dots (6)$$

where $$u_x' = (k_x u_x + k_y u_y)/k. \quad \dots (7)$$

Taking the *curl* of eq. (1) we have

$$\rho_0 \left[\frac{\nu}{k_1} + (1 + \alpha_0) \frac{\partial}{\partial t} \right] curl\, u + \nabla \rho_0 x \left[\frac{\nu}{k_1} + (1 + \alpha_0) \frac{\partial}{\partial t} \right] u$$

$$= -2\rho_0\, curl\, (\Omega x u) - 2\nabla \rho_0 x\, (\nabla x u) + g(t)\, \nabla \rho x e_z, \quad \dots (8)$$

where we have assumed that:

$$\frac{KN\tau_1}{\rho_0} v = \alpha_0 u. \quad \dots (9)$$

The quantity $KN\tau_1/\rho_0$ denotes mass concentration of suspended particles. Eq. (9) shows that the flow of suspended particles may be in the direction of the fluid flow or in the opposite direction depending on the sign of α_0, positive or negative, respectively.

The z component of (8) gives

$$\frac{\partial u_y'}{\partial t} = -2\Omega u_x', \quad \dots (10)$$

where $$u_y' = (k_x u_y - k_y u_x)/k; \quad \dots (11)$$

and k_x times the x component of (8), when added to k_y times the y component of (8), reduces to

$$\rho_0 \left\{ \left[\frac{\nu}{k_1} + (1 + \alpha_0) \frac{\partial}{\partial t} \right] \frac{\partial}{\partial z} u_y' + 2\Omega \frac{\partial u_x'}{\partial z} \right\} + \frac{\rho_0}{\lambda} \left\{ \left[\frac{\nu}{k_1} + (1 + \alpha_0) \frac{\partial}{\partial t} \right] u_y' + 2\Omega u_x' \right\} = 0 \quad \dots (12)$$

which, in view of (4), is identically satisfied. Also, k_y times the x component of (8), when subtracted from k_x times the y component of (8), gives

$$\left[\frac{\nu}{k} + (1 + \alpha_0) \frac{\partial}{\partial t} \right] \left[\frac{\partial u_x'}{\partial z} - iku_z + \frac{u_x'}{\lambda} \right] = 2\Omega \frac{\partial}{\partial z} u_y' + \frac{2\Omega}{\lambda} u_y' - ig \frac{\rho}{\rho_0} k. \quad \dots (13)$$

We express u_z in terms of $\frac{\rho}{\rho_0}$ from eq. (3) as

$$u_z = -\lambda \frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right). \quad \ldots (14)$$

Using (6), (10) and (14) we have u_x' and u_y' in terms of $\frac{\rho}{\rho_0}$.

Expressing u_x', u_y' and u_z in terms of $\frac{\rho}{\rho_0}$ we have from eq. (13),

$$\left[\frac{v}{k_1} + (1+\alpha_0)\frac{\partial}{\partial t}\right]\left[\frac{\partial^3}{\partial z^2 \partial t} + \frac{1}{\lambda}\frac{\partial^2}{\partial z \partial t} - k^2 \frac{\partial}{\partial t}\right]\left(\frac{\rho}{\rho_0}\right)$$

$$+ 4\Omega^2\left(\frac{\partial^2}{\partial z^2} + \frac{1}{\lambda}\frac{\partial}{\partial z}\right)\left(\frac{\rho}{\rho_0}\right) - \frac{k^2 g(t)}{\lambda}\left(\frac{\rho}{\rho_0}\right) = 0. \quad \ldots (15)$$

We consider a variable separable solution for $\frac{\rho}{\rho_0}$, so that we assume $\frac{\rho}{\rho_0}$ to be in the form

$$\frac{\rho}{\rho_0} = T(t\ Z(z). \quad \ldots (16)$$

Eq. (15) gives

$$\frac{\lambda v}{k^2 g k_1}\frac{T'}{T}\left[\frac{Z'' + \frac{Z'}{\lambda} - k^2 Z}{Z}\right] + \frac{\lambda}{k^2 g}\frac{T''}{T}(1+\alpha_0)$$

$$\left[\frac{Z'' + \frac{Z'}{\lambda} - k^2 Z}{Z}\right] = 1 - \frac{4\Omega^2\lambda}{k^2 g(t)}\frac{Z'' + \frac{Z'}{\lambda}}{Z}, \quad \ldots (17)$$

where '' represents differentiation with respect to the independent variable.

To solve (17) we take

$$\left(Z'' + \frac{Z'}{\lambda}\right)Z^{-1} = n_0, \quad \ldots (18)$$

where n_0 is a constant, and (17) reduces to

$$T'' + \frac{v}{k_1(1+\alpha_0)}T' - \left[\frac{k^2 g(t) - 4\lambda\Omega^2 n_0}{\lambda(1+\alpha_0)(n_0 - k^2)}\right]T = 0. \quad \ldots (19)$$

Putting

$$T(t) = W(t)\, e^{-\frac{v}{2k_1(1+\alpha_0)}t}, \quad \ldots (20)$$

then, eq. (19) reduces to

$$W''(t) - \left[\frac{\nu^2}{2k_1^2(1+\alpha_0)^2} + \frac{k^2 g(t) - 4\lambda\Omega^2 n_0}{\lambda(1+\alpha_0)(n_0 - k^2)} \right] W(t) = 0. \quad \text{... (21)}$$

Eq. (21) can be written in the form

$$W''(\tau) - (\gamma + \alpha + \beta\cos\tau)\, W(\tau) = 0, \quad \text{... (22)}$$

where,

$$\tau = \omega_s t, \quad \text{... (23)}$$

$$\alpha = (g_0 + 4\Omega^2\lambda n_0/k^2)\,(l_0\,\omega_s^2)^{-1}, \quad \text{... (24)}$$

$$l_0 = \lambda(1+\alpha_0)\,(k^2 - n_0)/k^2, \quad \text{... (25)}$$

$$\beta = \alpha/l_0 \quad \text{... (26)}$$

and

$$\gamma = \frac{\nu^2}{2k_1^2\,\omega_s^2\,(1+\alpha_0)^2}. \quad \text{... (27)}$$

In the absence of porous media and suspended particles, eq. (22) reduces to eq. (11) in Ref. [4]. Also, in the absence of rotation, porous media and suspended particles, eq. (22) reduces to eq. (3) in Ref. (3).

Since the normal component of fluid velocity vanishes on the rigid planes, the solution of (18) in view of (14), gives

$$n_0 = -\left(\frac{1}{4\lambda^2} + \frac{n^2\pi^2}{d^2} \right), \quad \text{... (28)}$$

where n is an integer (vertical wave number).

3. Discussion

3.1. In the Absence of Porous Media

3.1.a. The Effect of Rotation and Vertical Oscillations

In this case eq. (22) reduces to

$$\frac{d^2W}{d\tau^2} - (\alpha + \beta\cos\tau)\, w = 0, \quad \text{... (29)}$$

which is the Mathieu Equation, with α and β defined by eq. (24) and (26). It is clear that the β coordinate is unaffected by rotation. On the other hand the α coordinate, since no is negative, is decreased by rotation. The effect of rotation is thus to shift the point (α, β) horizontally to the left from its original position in the absence of rotation and hence towards the stable regions in the mathieu diagram[9&10].

3.1.b. The Effect of Suspended Particles

Comparing α and β in the absence of suspended particles with those in the presence of suspended particles, we find that $l_0 \gtrless l$ according to α_0 is positive or negative. This means that the suspended particles do not affect the regions of stability. On the other hand they have a stabilizing effect when α_0 is positive and a destabilizing effect when α_0 is negative.

3.2. In the Absence of Suspended Particles

Eq. (22) can be written in the form

$$\frac{d^2W}{d\tau^2} - (\alpha + \beta \cos \tau)\, W = 0, \qquad \text{... (30)}$$

where

$$\gamma = \frac{g_0}{\omega_s^2\, l} + \frac{v^2}{2k_1\omega_s^2}, \qquad \text{... (31)}$$

$$\beta = \frac{a}{l} \qquad \text{... (32)}$$

and

$$l = \lambda(k^2 - n_0)/k^2. \qquad \text{... (33)}$$

For $v \to 0$, eq. (30) reduces to eq. (3) in ref. [3] which describes dynamic stabilization on Rayleigh-Taylor Stability in the absence of finite Lamar radius effects.

The relation between γ and β can be written in the form

$$\beta = \frac{a\omega_s^2}{g_0}\gamma - \frac{av^2}{2k_1^2\, g_0}. \qquad \text{... (34)}$$

When $v = 0$ (i.e., in the absence of porous media), for a given value of $a\omega_s^2/g_0$ the possible values of γ and β lie on a straight line in the mathieu diagram[9]. The upper extreme point of this line can be made to fall inside the first stable region by an appropriate choice of a and ω_s. On the other hand, it is impossible to avoid the line passing through the lower unstable region.

When $v \neq 0$, equation (34) shows the relation between γ and β through a line which always passes through the unstable region. This means that the porous media has a destabilizing effect.

REFERENCES

1. L. Rayleigh, *Proc. London math. Soc.* **14** (1883) 170.
2. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Ch. X, Oxford Univ. Press, London, 1961.
3. J. Wesson, *Phys. Fluids.* **13** (1970) 761.

4. B. B. Chakraborty and M. Bandyopadhyay, *Phys. Fluids* **18** (1976) 762.
5. H. Alfven and P. Carlquist, *Astrophys. Space Sci.* **55** (1978) 487.
6. P. Kirti and M. Seema, *J. math. phys. Sci.* **28** (1994) 75.
7. J. W. Scanlon and L. A. Segel, *Phys. Fluids.* **16** (1973) 1573.
8. L. D. Landau and E. M. Lifshitz, *Mechanics* Pergamon Press, Ltd., London (1960) 126.
9. J. J. Stoker, *Nonlinear Vibration in Mechanical and Electrical Systems.* Wiley, New York, Chap VI (1950).
10. N. W. Mclachlan, *Theory and Application of Matheiu Functions* Oxford Univ. Press. London, England (1950), 40 and 98.