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INEQUALITIES CONNECTED WITH WEIGHTED MINIMAX SERIES

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Certain weighted minimax series are used in order to establish several properties connected with the best polynomial approximation of continuous functions. Specially, the existence of a certain sequence leads to define some Banach algebras.

Key Words : Polynomial Approximation Theory; Minimax Series.

1. INTRODUCTION

It is known that in order to investigate the connection between the errors concerning to best algebraic polynomial approximations

$$(E_n(f))_p = \inf (\| f - p \|_p;\ p = \text{algebraic polynomial of degree} \le n)$$

with $f \in L_p$ (D) and the norms (the same as its equivalents norms) of certain functional spaces in which $1 \le p \le \infty$, it is sometimes necessary to have recourse to the so-called moduli of smoothness. In this sense, some Besov-type spaces generated by these moduli are related to the sequence $\{(E_n(f))_p\}$.

One of the interesting results about Besov spaces is, e.g., among others that the norms of these spaces for periodic functions of period 2π (that is, $f \in L_p$ (T)) have an equivalent norm $\| f \|_{B^{p^*}_{\alpha, s}}$ which can be expressed in terms of such moduli of smoothness[4]. Other moduli of smoothness to investigate best algebraic polynomial approximation $E_n(f)_p$ can be seen in [3].

It must be remarked that, in particular, of $p = \infty$ and $s - 1$, this type of norms are defined by series involving weighted minimax series. In a more general framework the role of the equivalence between the Peetre K-functional and the moduli of smoothness leads to other important results in approximation processes[1&2].

The inequalities established in the present paper, related to the mentioned weighted minimax series allow to prove the existence of a certain sequence of constants connected with best algebraic polynomial approximation. On the other hand, these inequalities give rise to certain Banach algebras.

2. DEFINITIONS AND SOME CONSIDERATIONS

We start by recalling some basic definitions and refer to [5], [6], [7] for more details and references.

Let f be a continuous function on $[a, b]$. Π_n will denote the set of all polynomials of degree n at most and Π the set of all polynomials. For each n, the minimax of f is given by :

$$E_n(f) = \|f - p_n\|_\infty = \inf_{p \in \pi_n} \|f - p\|_\infty,$$

where p_n stands for the best approximation of f on π_n, in the uniform norm.

The expression

$$S(f) = \sum_{k=0}^{\infty} E_k(f)$$

is called minimax series of f and represents the sum of all the errors of the best polynomial approximations of f; hence, it can be seen as a "measure" of how good the function f can be approximated by polynomials.

In an earlier paper[6], the following inequality:

$$(fg) \leq KS(f)S(g),$$

where f and g denote continuous functions having zeros on $[a, b]$, $S(f) < \infty$ and $S(g) < \infty$ and K a certain positive constant, was established. This result leads us to consider weighted minimax series of the form :

$$N_r(f) = E_0(f) + \sum_{k=1}^{\infty} k^r E_k(f), \; r = 1, 2, \ldots.$$

Note that for $r = 0$ is $N_0(f) = S(f)$, and fix a point $x_0 \in [a, b]$ in order to assure that if $f(x_0) = 0$ and $E_0(f) = 0$, then $f = 0$.

The main result of this paper is that there exists a sequence (α_k) of positive numbers, independent of the point x_0, such that

$$N_k(fg) \leq \alpha_k N_k(f) \, N_k(g).$$

Finally using techniques with the moduli of smoothness is proved that this inequality is valid also for $r > 0$ arbitrary.

3. THE OPERATORS T AND U : PREVIOUS RESULTS

Let

$$S_0(f) = S(f), \; S_1(f) = \sum_{n=0}^{\infty} S(f - p_n), \text{ and } S_{j+1}(f) = \sum_{n=0}^{\infty} S_j(f - p_n), j = 1, 2, \ldots,$$

and the support spaces $F_j = \{f \in C[a,b]$ such that $f(x_0) = 0$ and $S_j(f) < \infty\}$.

Then S_j is a norm in F_j, for all $j = 0, 1, 2, \ldots$.

To prove our next result, we consider

Lemma 1 — (*a*) $E_k(f - p_n) = E_n(f)$ $(0 \le k \le n)$ and (b) $E_k(f - p_n) = E_k(f)$, $(k \ge n)$

PROOF : (*a*) If q_k is the best approximation of $f - p_n$ in π_k, one has:

$$E_n(f) \le \| f - p_n - q_k \|_\infty = E_k(f - p_n);$$

on the other hand

$$E_k(f - p_n) \le \| f - p_n \|_\infty = E_n(f).$$

(*b*) It follows taking into account that $p_n \in \pi_k$, $\forall\, k \ge n$. ■

We now define the following operators: The operator T, by means of the inequality:

$$T(A_0, A_1, \ldots, A_r) = A_0 N_0(f) + A_1 N_1(f) + \ldots + A_r N_r(f),\ A_0, A_1, \ldots, A_r \in I\!R^{r+1}$$

$$U(A_0, A_1, \ldots, A_r; n)$$

$$= A_0 \left(N_0(f) - \sum_{k<n} (E_k(f) - E_n(f)) \right) + A_1 \left(N_1(f) - \sum_{k<n} k(E_k(f) - E_n(f)) \right)$$

$$+ A_2 \left(N_2(f) - \sum_{k<n} k^2(E_k(f) - E_n(f)) \right) + \ldots + A_r \left(N_r(f) - \sum_{k<n} k^r(E_k(f) - E_n(f)) \right).$$

Proposition 1 — If $S_r(f) = T(A_0, A_1, \ldots, A_r)$ and p_n is the best approximation of f on π_n, one has:

$$S_r(f - p_n) = U(A_0, A_1, \ldots, A_r; n).$$

PROOF : Let $S_r(f) = A_0 N_0(f) + \ldots + A_r N_r(f)$.

Then

$$S(f - p_n) = A_0 N_0\,(f - p_n) + \ldots + A_r N_r\,(f - p_n)$$

$$= A_0 \left(S(f) - \sum_{k<n} (E_k(f) - E_n(f)) \right) + A_1 \left(\sum_{k=0}^{\infty} k E_k(f) - \sum_{k<n} k(E_k(f) - E_n(f)) \right)$$

$$+ \dots + A_r \left(\sum_{k=0}^{\infty} k^r E_k(f) - \sum_{k<n} k^r (E_k(f) - E_n(f)) \right)$$

$$= A_0 \left(N_0(f) - \sum_{k<n} (E_k(f) - E_n(f)) \right) + A_1 \left(N_1(f) \sum_{k<n}^{\infty} kE_k(f) - \sum_{k<n} k(E_k(f) - E_n(f)) \right)$$

$$+ \dots + A_r \left(N_r(f) - \sum_{k<n}^{\infty} k^r E_k(f) - E_n(f)) \right) = U(A_0, A_1, \dots, A_r; n) \qquad \blacksquare$$

Theorem 1 — *If* $S_{r-1}(f) = T(A_0, A_1, \dots, A_{r-1})$, *then*

$$S_r(f) = T(B_0, B_1, \dots, B_r),$$

where

$$B_0 = A_0 + A_{r-1}\, \alpha_0^{(r-1)}, B_1 = A_0 + A_{r-1}\, \alpha_1^{(r-1)}, \dots$$

$$B_{r-2} = A_0 + A_{r-1}\, \alpha_{r-2}^{(r-1)}, B_{r-1} = A_0 + A_{r-1} \left(\alpha_{r-1}^{(r-1)} + 1 \right), \text{ and } B_r = 2A_{r-1},$$

where

$$1^r + 2^4 + \dots + (n-1)^r = n^{r+1} + \alpha_r^{(r)} + \alpha_{r-1}^{(r)} n^{r-1} + \dots + \alpha_0^{(r)}.$$

PROOF : The proof follows by induction on r.
Thus, for $r = 1$;

$$S(f) = \sum_{n=0}^{\infty} E_n(f) = T(1),$$

$$S(f - p_n) = U(1; n),$$

and

$$\sum_{n=0}^{\infty} U(1; n) = \sum_{n=0}^{\infty} \left(S(f) - \sum_{k<n} (E_k(f) - E_n(f)) \right) = \sum_{n=0}^{\infty} E_n(f) + 2 \sum_{n=0}^{\infty} nE_n(f) = T(1, 2).$$

$$B_0 = A_0 + A_0\, (\alpha_0^{(0)} + 1)$$

$$B_1 = 2A_0.$$

$$1^0 + 2^0 + 3^0 + \ldots + (n-1)^0 = 1 + 1 = \ldots + 1 = (n-1).$$

Hence, $\alpha_0^{(0)} = -1, A_0 = 1, B_0 = 1,$ and $B_1 = 2.$

Suppose that this is true for $r = m$. Then, for $r = m + 1$, we obtain

$$\sum_{n=0}^{\infty} U(A_0, A_1, \ldots, A_m; n) = \sum_{n=0}^{\infty} \Big[U(A_0, A_1, \ldots, A_{m-1}; n)$$

$$+ A_m(N_m(f) - \sum_{k<n} k^m (E_k(f) - E_n(f))) \Big],$$

and by the induction hypothesis,

$$\sum_{n=0}^{\infty} U(A_0, A_1, \ldots, A_{m-1}; n) = T(C_0, C_1, \ldots, C_{m-1}, C_m),$$

where $C_0 = A_0 + A_{m-1}\,\alpha_0^{(m-1)},\ C_1 = A_0 + A_{m-1}\,\alpha_1^{(m-1)}, \ldots, C_{m-2} = A_0 + A_{m-1}\,\alpha_{m-2}^{(m-1)},$

$$C_{m-1} = A_0 + A_{m-1}\left(\alpha_{m-1}^{(m-1)} + 1\right) \text{ and } C_m = 2A_{m-1}.$$

On the other hand,

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} k^m E_k(f) - \sum_{k<n} k^m (E_k(f) - E_n(f)) \right)$$

$$= 1^m E_1(f) + 2.2^m E_2(f) + 3.3^m E_3(f) + \sum_{n=0}^{\infty} [1^m + 2^m + \ldots + (n-1)^m] E_n(f)$$

$$= \sum_{n=0}^{\infty} n^{m+1} E_n(f) + \sum_{n=0}^{\infty} [1^m + 2^m + \ldots (n-1)^m] E_n(f).$$

Hence,

$$\sum_{n=0}^{\infty} U(A_0, A_1, \ldots, A_m; n) = T(C_0, C_1, \ldots, C_{m-1}, C_m)$$

$$+ A_m \left(\sum_{n=0}^{\infty} n^m E_n(f) + \sum_{n=0}^{\infty} n^{m+1} E_n(f) + \sum_{n=0}^{\infty} [1^m + 2^m + \ldots (n-1)^m] E_n(f) \right)$$

$$= A_m \left(\sum_{n=0}^{\infty} n^m E_n(f) + \sum_{n=0}^{\infty} n^{m+1} E_n(f) + \alpha_0^{(m)} \sum_{n=0}^{\infty} E_n(f) + \right.$$

$$\left. + \alpha_1^{(m)} \sum_{n=0}^{\infty} nE_n(f) + \ldots + \alpha_n^{(m)} \sum_{n=0}^{\infty} n^m E_n(f) + \sum_{n=0}^{\infty} n^{m+1} E_n(f) \right)$$

$$= T(C_0, C_1, \ldots, C_{m-1}, C_m) + A_m T\left(\alpha_0^{(m)}, \alpha_1^{(m)}, \alpha_m^{(m)} + 1, 2 \right)$$

$$= T(C_0 + A_m \alpha_0^{(m)}, C_1 + A_m \alpha_1^{(m)}, \ldots, C_{m-1} + A_m \alpha_{m-1}^{(m)}, C_m + A_m (\alpha_m^{(m)} + 1), 2A_m)$$

$$= T(B_0, B_1, \ldots, B_{m-1}, B_m, B_{m+1}),$$

where

$$B_0 = A_0 + A_m \alpha_0^{(m)}, B_1 = A_0 + A_m \alpha_1^{(m)}, \ldots, B_{m-1} = A_0 + A_m \alpha_m^{(m)},$$

$$B_m = A_0 + A_m (\alpha_m^{(m)} + 1) \text{ and } B_{m+1} = 2A_m$$ ■

Corollary 1 — There exist constants $A_0, A_1, \ldots, A_j$ such that

$$S_j(f) = A_0 N_0(f) + A_1 N_1(f) + \ldots + A_j N_j(f), \quad \forall f \in F_j.$$

PROOF : The results follows from theorem 1 and taking into account that

$$S(f) = S_0(f) = T(1)$$ ■

Corollary 2 — If p_n is the best approximation of $f \in F_j$ on π_n, one has

$$S_j(f - p_n) \leq S_j(f), \quad \forall j = 0, 1, 2, \ldots$$

and

$$S_j(p_n) \leq 2S_j(f), \qquad \forall j = 0, 1, 2, \ldots .$$

PROOF : $S_j(f - p_n) = U(A_0, A_1, \ldots, A_j; n) \leq A_0 N_0(f) + A_1 N_1(f) + \ldots + A_j N_j(f) = S_j(f),$

$\forall j = 0, 1, 2, \ldots$.

Also

$$S_j(p_n) \leq S_j(f - p_n) + S_j(f) \leq 2S_j(f), \; j = 0, 1, 2, \ldots$$ ■

Theorem 2 — $S_j(f - p_n) \leq S_j(f - p), \;\; \forall p \in \pi_n.$

PROOF : $S_j(f-p_n) = A_0N_0(f-p_n) + A_1N_1(f-p_n) + \ldots + A_jN_j(f-p_n)$

$$\leq A_0N_0(f-p) + A_1N_1(f-p) + \ldots + A_jN_j(f-p), \ \forall\ p \in \pi_n$$ ■

***Theorem* 3** — *The norms N and S_j are equivalents in F_j for all j = 0, 1, 2, 3.*

PROOF : $S_j(f) = A_0N_0(f) + A_1N_1(f) + \ldots + A_jN_j(f)$,

and

$$N_0(f) \leq N_1(f) \leq \ldots \leq N_j(f).$$

Then

$$N_j(f) \leq S_j(f) \leq (A_0 + \ldots + A_j)\ N_j(f)$$ ■

Corollary 3 — $S_j(f-p_n) \leq S_j(f-p), \ \forall\ p \in \pi_n$.

PROOF : $S_j(f-p_n) = A_0N_0(f-p_n) + A_1N_1(f-p_n) + \ldots + A_jN_j(f-p_n)$

$$\leq A_0N_0(f-p) + A_1N_1(f-p) + \ldots + A_jN_j(f-p), \ \forall\ p \in \pi_n$$ ■

This corollary establishes that if p_n is the best polynomial approximation of f (with the uniform norm) in π_n, then p_n is also a best approximation of f with the norm S_j.

4. MAIN RESULTS

Lemma 2 — If f has a zero in $[a, b]$, then $\|f\|_\infty \leq 2S(f)$.

PROOF : There exists a constant c such that

$$\|f-c\|_\infty = E_0(f) \leq S(f),$$

and there exists $x_0 \in [a, b]$ such that $f(x_0) = 0$. Hence

$$|c| \leq \|f-c\|_\infty = E_0(f).$$

Then

$$\|f\|_\infty \leq \|f-c\|_\infty + |c| \leq E_0(f) = 2E_0\,(f) \leq 2S(f)$$ ■

***Theorem* 4** — *If f and g have a zero on $[a, b]$, then :*

$$S(fg) \leq \frac{3}{2}\,[\|f\|_g\, S(g) + \|g\|_\infty\, S(f)] + 2S(f)\, S(g)$$

PROOF : Note that

$$E_{r+s}(fg) \leq \| fg - p_r q_s \|_\infty \leq \| fg - fq_s + fq_s - p_r q_s \|_\infty$$

$$= \| f(g - q_s) + q_s(f - p_r) \|_\infty \leq \| f \|_\infty E_s(g) + \| q_s \|_\infty E_r(f).$$

On the other hand

$$\| q_s \|_\infty \leq S(g) + \| g \|_\infty,$$

Thus,

$$E_{r+s}(fg) \leq \| f \|_\infty E_s(g) + S(g) E_r(f) + \| g \|_\infty E_r(f).$$

Now, for $r = s + 2$, we get:

$$\sum_{s=0}^{\infty} E_{2s+2}(fg) \leq \| f \|_\infty S(g) = (S(g) + \| g \|_\infty)(S(f) - E_0(f) - E_1(f)) \quad \text{...(4.1)}$$

and for $r = s + 1$,

$$\sum_{s=0}^{\infty} E_{2s+2}(fg) \leq \| f \|_\infty S(g) = (S(g) + \| g \|_\infty)(S(f) - E_0(f)) \quad \text{... (4.2)}$$

From (4.1) and (4.2) it follows that

$$S(fg) = E_0(fg) + \sum_{s=0}^{\infty} E_{2s+1}(fg) + \sum_{s=0}^{\infty} E_{2s+2}(fg)$$

$$\leq \| f \|_\infty \| g \|_\infty + 2 \| f \|_\infty S(g) + 2(S(g) + \| g \|_\infty)(s(f) - E_0(f)) - (S(g) + \| g \|_\infty) E_0(f)$$

$$\leq \| f \|_\infty \| g \|_\infty + 2 \| f \|_g S(g) + 2S(f) S(g) + 2 \| g \|_\infty S(f) - 2(S(g) + \| g \|_\infty E_0(f).$$

and by involving lemma 2,

$$S(fg) \leq \| f \|_g \| g \|_\infty + 2 \| f \|_\infty S(g) + 2S(f)S(g) + 2 \| g \|_\infty S(f) - 2(s(g) + \| g \|_\infty) \frac{\| f \|_\infty}{2}$$

$$= \| f \|_\infty \| g \|_\infty + 2 \| f \|_g S(g) + 2S(f) S(g) + 2 \| g \|_\infty S(f) - \| f \|_g S(g) - \| f \|_\infty \| g \|_\infty$$

$$= \| f \|_\infty S(g) + 2 \| g \|_\infty S(f) + 2S(f) S(g).$$

Interchanging f and g, one has:

$$S(fg) \leq \| g \|_\infty S(f) + 2 \| f \|_\infty S(g) + 2S(f)S(g).$$

Then

$$S(fg) \leq \frac{3}{2}(\| f \|_\infty S(g) + \| g \|_\infty (S(f)) + 2S(f)\, S(g) \qquad \blacksquare$$

Corollary 4 — If f and g have a zero on $[a, b]$, then

$$S(fg) \leq 8S(f)S(g),$$

PROOF : Taking into account that

$$\| f \|_\infty \leq 2S(f) \text{ and } \| g \|_\infty \leq 2S(g),$$

it follows

$$S(fg) \leq \frac{3}{2}(2S(f)\, S(g) + 2S(g)S(f)) + 2S(f)S(g) = 8S(f)S(g) \qquad \blacksquare$$

Theorem 5 — *$S_j(fg) \leq k_j S_j(f) S_j(g)$, for all f and $g \in F_j$ with $k_j = 6k_{j-1}$.*

PROOF : Let p_n and q_n the best uniform approximates of f and g on π_n, respectively. Then

$$S_{j-1,2n}(fg) \leq S_{j-1}(fg - p_n q_n) = S_{j-1}(fg - fq_n + fq_n - p_n q_n)$$

$$\leq S_{j-1}(f(g - q_n)) + S_{j-1}((f - p_n)q_n)$$

$$\leq K_{j-1} S_{j-1}(f) S_{j-1}(g - q_n) + K_{j-1} S_{j-1}(f - p_n) S_{j-1}(q_n). \qquad \text{... (4.3)}$$

Note that $S_{j-1}(f) = S_{j-1}(f - p_0(f))$ is the first term of the series $S_j(f)$, hence

$$S_{j-1}(f) \leq S_j(f).$$

Also, by corollary 2

$$S_{j-1}(q_n) \leq 2S_{j-1}(g).$$

Hence

$$S_{j-1,2n}(fg) \leq K_{j-1} S_j(f) S_{j-1}(g - g_n) + 2K_{j-1} S_{j-1}(g) S_{j-1}(f - p_n).$$

Thus

$$\sum_{n=0}^{\infty} S_{j-i,2n}(fg) \leq K_{j-1} S_{j-1}(f) \sum_{n=0}^{\infty} S_{j-1}(g - g_n) + 2K_{j-1} S_{j-1}(g) \sum_{n=0}^{\infty} S_{j-1}(f - p_n)$$

$$= K_{j-1} S_{j-1}(f) S_j(g) + 2K_{j-1} S_{j-1}(g) S_j(f)$$

$$\leq K_{j-1}\, S_j(f)\, S_j(g) + 2K_{j-1}\, S_j(g)\, S_j(f) = 3S_j(f)\, S_j(g).$$

On the other hand

$$\sum_{n=1}^{\infty} S_{j-1,\,2n-1}(fg) \leq \sum_{n=1}^{\infty} S_{j-1,\,2n}(fg).$$

$$\sum_{n=0}^{\infty} S_{j-1,\,n}\ (fg) = \sum_{n=1}^{\infty} S_{j-1,\,2n-1}(fg) + \sum_{n=1}^{\infty} S_{j-1,\,2n}(fg)$$

$$\leq 6K_{j-1}\, S_j(f) S_j(g) = K_j S_j(f) S_j(g), \text{ with } K_j = 6K_{j-1}$$ ■

Remark 1 : If in (4.3) we use the boundedness

$$S_{j-1}(q_n) \leq S_{j-1}(q_n - g) = S_{j-1}(g),$$

we get

$$S_{j,2n}(fg) \leq K_{j-1}\, S_{j-1}(f) S_{j-1}(g-g_n) + K_{j-1}\, S_{j-1}(f-p_n)\, S_{j-1}(q_n)$$

$$\leq K_{j-1}\, S_{j-1}(f)\, S_{j-1}(g-g_n) + K_{j-1}\, S_{j-1}(f-p_n)\, [S_{j-1}(q_n-g) + S_{j-1}(g)].$$

Hence,

$$\sum_{n=0}^{\infty} S_{j-1,\,2n} fg) \leq K_{j-1}\, S_{j-1}(f) S_j(g) + K_{j-1}\, S_j(f) S_{j-1}(g)$$

$$+\ K_{j-1} \sum_{n=0}^{\infty} S_{j-1,\,2n}(f-p_n)\, S_{j-1,\,2n}(g-q_n)$$

$$\leq 2\, K_{j-1}\, S_j(f) S_j(g) + K_{j-1} S_j(f)\, S_j(g) = 3K_{j-1} S_j(f) S_j(g).$$

Then

$$\sum_{n=0}^{\infty} S_{j-1,\,2n}(fg) \leq 6K_{j-1}\, S_j(f)\, S_j(g),$$

And the same boundedness is obtained.

Remark 2 : The assertion of theorem 5 holds whenever f and g have zeros on $[a, b]$ and $S_j(f) < \infty$, $S_j(g) < \infty$.

Remark 2 : The assertion of theorem 5 holds whenever f and g have zeros on $[a, b]$ and $S_j(f) < \infty$, $S_j(g) < \infty$.

Theorem 6 — *There are constants* $\alpha_k > 0$ *such that*

$$N_k(fg) \leq \alpha_k N_k(f) N_k(g) \quad \forall f, g \in F_k. \qquad \text{... (4.4)}$$

PROOF : The norm N_k is equivalent to S_k. That is, there exist constants $m_k > 0, M_k > 0$ such that

$$m_k S_k(f) \leq N_k(f) \leq M_k S_k(f), \quad \forall f, \in F_k.$$

Then

$$m_k S_k(fg) \leq N_k(fg) \leq N_k S_k(fg),$$

and

$$S_k(f) \leq \frac{1}{m_k} N_k(f), \quad \forall f, \in F_k.$$

Consequently:

$$N_k(fg) \leq M_k S_k(fg) \leq M_k K_k S_k(f)\, S_k(g) \leq \frac{M_k K_k}{m_k^2} N_k(f) N_k(g) = \alpha_k N_k(f) N_k(g),$$

where

$$\alpha_k = \frac{M_k K_k}{m_k^2}. \qquad \blacksquare$$

5. THE SEQUENCE $\{\gamma_k\}$

The sequence $\left\{ \gamma_k \right\}_{k=0}^{\infty}$ given by

$$\gamma_k = \inf \{\alpha_k : N_k(fg) \leq \alpha_k N_k(f) N_k(g), \ \forall f, g \in C[a, b],$$

f and g with zeros in $[a, b]$ and $N_k(f) < \infty, N_k(g) < \infty\}$.

is a unique sequence, completely determined, and stands for a sequence of numbers associated to the polynomial approximation. Clearly

$$\gamma_k = \text{Sup} \left(\frac{N_k(fg)}{N_k(f) N_k(g)} : f \text{ and } g \in C[a, b], \text{ with zeros in } [a, b], \right.$$

$N_k(f) < \infty$ and $N_k(g) < \infty\}$.

It follows from corollary 4:

Corollary 5 — $\gamma_0 \leq 8$.

PROOF : $N_0(fg) = S(fg) \leq 8S(f)S(g) = 8N_0(f)N_0(g)$ ■

Note 1 : What is the exact value of γ_0?. This is an open question.

6. THE BANACH ALGEBRA (C_k, N_k)

The vectorial spaces constituted by the set of functions

$$C_k = \{f \in C[a, b] : f(x_0) = 0, N_k(f) < \infty\}$$

with

$$N_k(f) = E_0(f) + \sum_{n=1}^{\infty} n^k E_n(f), \ k = 1, 2, \ldots$$

are, as can easily be seen, Banach spaces.

Furthermore, theorem 6 allows to assert that these spaces with the punctual product $fog = \frac{fg}{\alpha_k}$ constitute Banach algebras. In the particular case, $k = 0$, the Banach Algebra (C_0, S) in connection with minimax series is studied in [8].

Note that if we define $M_r(f) = \sum_{k=0}^{\infty} (k+1)^r E^k(f), k > r+1, N_0 = S, r > 0$

arbitrary (not only integer), taking into account the following equivalence proved in [4], one has:

$$M_r(F) = \sum_{k=0}^{\infty} (k+1)^r E_k(F) + \int_0^{1/m} \frac{w_\varphi^k(F, t)}{t^{r+2}} dt = \| F \|_{r+1}, \qquad \ldots (6.1)$$

where $$w_\varphi^k(fg, t) \leq 2^k \| f \|_\infty w_\varphi^k(g) + 2^k \| g \|_\infty w_\varphi^k(f),$$

and substituting in (6.1) we have

$$M_r(fg) \sim \| fg \|_{r+1} \leq 2^{k+1} \| f \|_{r+1} \| g \|_{r+1} \sim M_r(f) M_r(g).$$

Note also that the norms N_r and M_r are equivalents. In this sense, this result is a simple proof of theorem 6 which is valid also for $r > 0$ arbitrary, not only integer.

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