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SOME INTEGRAL INEQUALITIES OF GRÜSS TYPE

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Some classical and new integral inequalities of Grüss type are presented.

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1. GRÜSS INTEGRAL INEQUALITY

In 1935, G. Grüss, proved the following integral inequality which gives an estimation for the integral of a product in terms of the product of integrals (see for example [1, p. 296])

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)\, dx \right|$$

$$\leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma);$$

provided that f and g are two integrable functions on $[a, b]$ and satisfying the condition

$$\varphi \leq f(x) \leq \Phi, \quad \gamma \leq g(x) \leq \Gamma \qquad \dots (1.1)$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible and is achieved for $f(x) = g(x) = sgn\left(x - \frac{a+b}{2}\right)$.

Theorem 1.1 — *Let f and g be two functions defined and integrable on $[a, b]$. If (1.1) holds for each $x \in [a, b]$, where $\varphi, \Phi, \gamma, \Gamma$ are given real constants, and $h : [a, b] \to [0, \infty)$ is integrable and $\int_a^b h(x)dx > 0$, then*

$$\left| \int_a^b h(x)dx \cdot \int_a^b f(x)g(x)h(x)dx - \int_a^b f(x)h(x)dx \cdot \int_a^b g(x)h(x)dx \right|$$

$$\leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma)\left(\int_a^b h(x)dx \right)^2 \quad \text{... (1.2)}$$

and the constant $\frac{1}{4}$ is the best possible.

For the sake of completeness we give here a simple proof of this fact which is similar with the classical one for unweighted case (compare with [1, p. 296]).

Let us note that the following equality is valid :

$$\frac{1}{\int_a^b h(x)\,dx} \int_a^b f(x)g(x)h(x)dx - \frac{1}{\int_a^b h(x)dx} \int_a^b f(x)h(x)dx \cdot \frac{1}{\int_a^b h(x)dx} \int_a^b g(x)h(x)dx$$

$$= \frac{1}{2\left(\int_a^b h(x)dx \right)^2} \int_a^b \int_a^b (f(x) - f(y))\,(g(x) - g(y))\,h(x)h(y)dxdy. \quad \text{... (1.3)}$$

Applying Cauchy-Buniakowski-Schwarz's integral inequality for double integrals we have

$$\left[\frac{1}{2\left(\int_a^b h(x)dx \right)^2} \int_a^b \int_a^b (f(x) - f(y))\,(g(x) - g(y))\,h(x)h(y)dxdy \right]^2$$

$$\leq \frac{1}{2\left(\int_a^b h(x)dx \right)^2} \int_a^b \int_a^b (f(x) - f(y))^2\,h(x)h(y)dxdy$$

$$\times \frac{1}{2\left(\int_a^b h(x)dx \right)^2} \int_a^b \int_a^b (g(x) - g(y))^2\,h(x)h(y)dxdy$$

$$= \left[\frac{1}{\int_a^b h(x)dx} \int_a^b f^2(x)h(x)dx - \left(\frac{1}{\int_a^b h(x)dx} \int_a^b f(x)h(x)dx \right)^2 \right]$$

$$\times \left[\frac{1}{\int_a^b h(x)dx} \int_a^b g^2(x)h(x)dx - \left(\frac{1}{\int_a^b h(x)dx} \int_a^b g(x)h(x)dx \right)^2 \right]. \qquad \text{... (1.4)}$$

The following equality also holds

$$\frac{1}{\int_a^b h(x)dx} \int_a^b f^2(x)h(x)dx - \left(\frac{1}{\int_a^b h(x)dx} \int_a^b f(x)h(x)dx \right)^2$$

$$= \left(\Phi - \frac{1}{\int_a^b h(x)dx} \int_a^b f(x)h(x)dx \right) \cdot \left(\frac{1}{\int_a^b h(x)dx} \int_a^b f(x)h(x)dx - \varphi \right)$$

$$- \frac{1}{\int_a^b h(x)dx} \int_a^b (\Phi - f(x))\,(f(x) - \varphi)\,h(x)dx.$$

As, $(\Phi - f(x))\,(f(x) - \varphi) \geq 0$ for each $x \in [a, b]$, then

$$\frac{1}{\int_a^b h(x)dx} \int_a^b f^2(x)h(x)dx - \left(\frac{1}{\int_a^b h(x)\,dx} \int_a^b f(x)h(x)dx \right)^2$$

$$\leq \left(\Phi - \frac{1}{\int_a^b h(x)dx} \int_a^b f(x)h(x)dx \right) \cdot \left(\frac{1}{\int_a^b h(x)dx} \int_a^b f(x)h(x)dx - \varphi \right). \qquad \text{... (1.5)}$$

Similarly, we have

$$\frac{1}{\int_a^b h(x)dx}\int_a^b g^2(x)h(x)dx-\left(\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)h(x)\,dx\right)^2$$

$$\leq\left(\Gamma-\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)h(x)dx\right)\cdot\left(\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)h(x)dx-\gamma\right). \quad \text{... (1.6)}$$

Now, by (1.3), (1.4), (1.5) and (1.6) we get

$$\left|\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)g(x)h(x)dx-\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx\cdot\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)h(x)dx\right|$$

$$\leq\left(\Phi-\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx\right)\cdot\left(\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx-\varphi\right)$$

$$\times\left(\Gamma-\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)h(x)dx\right)\cdot\left(\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)h(x)dx-\gamma\right). \quad \text{... (1.7)}$$

Using the elementary inequality for real numbers:

$$4pq\leq(p+q)^2, p, q\in \boldsymbol{R}$$

we can state

$$4\left(\Phi-\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx\right)\cdot\left(\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx-\varphi\right) \quad \text{... (1.8)}$$

$$\leq(\Phi-\varphi)^2$$

and

$$4\left(\Gamma-\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)h(x)dx\right)\cdot\left(\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)h(x)dx-\gamma\right) \quad \text{... (1.9)}$$

$$\le (\Gamma-\gamma)^2.$$

Now, combining (1.7) with (1.8) and (1.9) we deduce the desired inequality (1.2). To prove the sharpness of (1.2), let choose $h(x) = 1$,

$$f(x) = g(x) = sgn\left(x-\frac{a+b}{2}\right) \text{ for all } x \in [a, b].$$

Then

$$\frac{1}{b-a}\int_a^b f(x)dx = 1,$$

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}\int_a^b g(x)dx = 0,$$

$$\Phi-\varphi = \Gamma-\gamma = 2$$

and the equality in (1.2) is realized. ■

For other inequalities of Grüss type see the book[1], where many other references are given.

2. The Case When Both Mappings are Lipschitzian

The following inequality of Grüss type for Lipschitzian mappings holds:

***Theorem* 2.1** — *Let* $f, g : [a, b] \to \mathbf{R}$ *be two lipschitzian mapings with the constants* $L_1 > 0$ *and* $L_2 > 0$, *i.e.,*

$$|f(x)-f(y)| \le L_1 |x-y|, \quad |g(x)-g(y)| \le L_2 |x-y| \quad \text{... (2.1)}$$

for all $x, y \in [a, b]$. *If* $p : [a, b] \to [0, \infty)$ *is integrable, then*

$$\left|\int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx\right|$$

$$\le L_1L_2\left[\int_a^b p(x)dx \cdot \int_a^b p(x)x^2dx - \left(\int_a^b p(x)xdx\right)^2\right] \quad \text{... (2.2)}$$

and the inequality is sharp.

PROOF : By (2.1) we have that

$$|f(x)-f(y))(g(x)-g(y))| \le L_1L_2(x-y)^2$$

for all $x, y \in [a, b]$.

Multiplying by $p(x)p(y) \ge 0$ and integrating on $[a, b]^2$, we get

$$\left| \int_a^b \int_a^b p(x)p(y)(f(x)-f(y))(g(x)-g(y))\,dxdy \right|$$

$$\le \int_a^b \int_a^b p(x)p(y)\,|(f(x)-f(y))(g(x)-g(y))|\,dxdy$$

$$\le L_1L_2 \int_a^b \int_a^b p(x)p(y)(x-y)^2\,dxdy.$$

As it is easy to see that

$$\frac{1}{2}\int_a^b \int_a^b (f(x)-f(y))(g(x)-g(y))\,p(x)p(y)\,dxdy.$$

$$= \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx$$

and

$$\frac{1}{2}\int_a^b \int_a^b p(x)p(y)(x-y)^2\,dxdy = \int_a^b p(x)dx \int_a^b p(x)x^2dx - \left(\int_a^b p(x)xdx\right)^2$$

the inequality (2.2) is thus obtained.

Now, if we chose $f(x) = L_1x$, $g(x) = L_2x$, then f is L_1-lipschitzian, g is L_2 -lipschitzian and the equality in (2.2) is realized for any p as above. ■

Corollary 2.2 — Under the above assumptions, we have

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)dx \right|$$

$$\le \frac{L_1L_2(b-a)^2}{12}. \qquad \text{... (2.3)}$$

The constant $\frac{1}{12}$ is the best possible.

We note that the above corollary is a natural generalization of a well-known result by Cebyšev (see for example [1, p. 297]):

Corollary 2.3 — Let $f, g : [a, b] \to \mathbf{R}$ be two differentiable mappings whose derivatives are bounded on (a, b). Denote $\| f' \|_\infty = \sup_{t \in (a, b)} | f'(t) | < \infty$. Then we have the inequality :

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right|$$

$$\leq \frac{\| f' \|_\infty \| g' \|_\infty}{12} (b-a)^2. \qquad \text{... (2.4)}$$

The constant $\frac{1}{12}$ is the best possible.

3. THE CASE WHEN f IS LIPSCHITZIAN

We are able to now to prove another inequality of Grüss type assuming that only one mapping is lipschitzian as follows :

Theorem 3.1 — *Let $f : [a, b] \to \mathbf{R}$ be a M-lipschitzian mapping on [a, b]. Then we have the inequality* :

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right|$$

$$\leq \begin{cases} M \| g \|_1 \text{ provided that } g \in L_1 [a, b] \\ M \left[\dfrac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} (b-a)^{\frac{1}{p}} \| g \|_q \text{ provided that } g \in L_q [a, b] \\ M \dfrac{(b-a)^3}{3} \| g \|_\infty \text{ provided that } g \in L_\infty [a, b] \end{cases} \qquad \text{... (3.1)}$$

PROOF : We have that

$$| f(x)g(y) - f(y)g(y) | \leq M | x - y | | g(y) |$$

for all $x, y \in [a, b]$, from where, by integration on $[a, b]^2$, we get that

$$\left| \int_a^b \int_a^b (f(x)g(y) - f(y)g(y)) \, dxdy \right| \leq M \int_a^b \int_a^b | x - y | | g(y) | \, dxdy.$$

But

$$\int_a^b \int_a^b (f(x)g(y) - f(y)g(y))\, dxdy = \int_a^b f(x)dx \int_a^b g(x)dx - (b-a) \int_a^b f(x)g(x)dx.$$

Now, if $g \in L_1\,[a, b]$, then

$$\int_a^b \int_a^b |x-y|\,|g(y)|\,dxdy \le (b-a) \max_{(x,y)\in[a,b]^2} |x-y| \int_a^b |g(y)|\,dy = (b-a)^2 \|g\|_1.$$

Now, assume that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1, g \in L_q\,[a, b]$. Then by Hölder's integral inequality we have:

$$\int_a^b \int_a^b |x-y|\,|g(y)|\,dxdy$$

$$\le \left(\int_a^b \int_a^b |x-y|^p\, dxdy \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b |g(y)|^q\, dxdy \right)^{\frac{1}{q}} = K^{\frac{1}{p}} (b-a)^{\frac{1}{q}} \|g\|_q$$

where

$$K := \int_a^b \int_a^b |x-y|^p\, dxdy = \int_a^b \left(\int_a^b |y-x|^p\, dy \right)^{dx}$$

$$= \int_a^b \left(\int_a^x |x-y|^p\, dy + \int_x^b |y-x|^p\, dy \right) dx$$

$$= \int_a^b \left[\frac{(x-a)^{p+1} + (b-x)^{p+1}}{p+1} \right] dx = \frac{2(b-a)^{p+2}}{(p+1)(p+2)}$$

and then we get

$$\int_a^b \int_a^b |x-y|\,|g(y)|\,dxdy \le \left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} (b-a)^{2+\frac{1}{p}} \|g\|_q.$$

Finally, assuming that $g \in L_\infty\,[a, b]$, we have that

$$\int_a^b \int_a^b |x-y|\,|g(y)|\,dxdy \le \|g\|_\infty \int_a^b \int_a^b |x-y|\,dxdy = \frac{(b-a)^3}{3} \|g\|_\infty.$$

The theorem is thus proved. ■

The following corollary is important in applications.

Corollary 3.2 — Let $f: [a, b] \to \mathbf{R}$ be a differentiable mapping whose derivative is bounded on (a, b). Then we have the inequality:

$$\left| \frac{1}{b-1} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right|$$

$$\leq \begin{cases} \| f' \|_\infty \| g \|_1 \text{ provided that } g \in L_1 [a, b] \\ \left[\dfrac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} (b-a)^{\frac{1}{p}} \| f' \|_\infty \| g \|_q \text{ provided that } g \in L_q [a, b] \\ \qquad p > 1, \dfrac{1}{p} + \dfrac{1}{1} = 1; \\ M \dfrac{(b-a)^3}{3} \| g \|_\infty \text{ provided that } g \in L_\infty [a, b] \end{cases} \quad \text{... (3.2)}$$

4. THE CASE WHEN f IS M–g-LIPSCHITZIAN

Another generalization of Grüss' integral inequality is embodied in the following theorem:

Theorem 4.1 — *Let $f, g : [a, b] \to \mathbf{R}$ be two integrable mappings on $[a, b]$ such that*

$$| f(x) - f(y) | \leq M | g(x) - g(y) | \quad \textit{for all } x, y \in [a, b]. \quad \text{... (4.1)}$$

Then we have the inequality:

$$\left| \int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx \right|$$

$$\leq M \left[\int_a^b p(x)dx \int_a^b p(x)g^2(x)dx - \left(\int_a^b p(x)g(x)dx \right)^2 \right] \quad \text{... (4.2)}$$

where $p ; [a, b] \to [0, \infty)$ is an arbitrary integrable function on $[a, b]$. The inequality (4.2) is sharp.

PROOF : By condition (4.1) we have

$$| (f(x) - f(y)) (g(x) - g(y)) | \leq M (g(x) - g(y))^2 \text{ for all } x, y \in [a, b].$$

Multiplying by $p(x)\, p(y) \geq 0$ and integrating on $[a, b]^2$ we get

$$\left| \int_a^b \int_a^b p(x)p(y) (f(x) - f(y)) (g(x) - g(y))\, dxdy \right|$$

$$\leq \int_a^b \int_a^b p(x)p(y) \mid (f(x) - f(y)) (g(x) - g(y)) \mid dxdy$$

$$\leq M \int_a^b \int_a^b p(x)p(y)(g(x) - g(y))^2 \, dxdy$$

which is clearly equivalent to (4.2).

Now, if we choose $f(x) = Mx$, $g(x) = x$, then the equality in the above inequality is realized for any p as above. ■

The following corollary is important for applications.

Corollary 4.2 — Let $f, g : [a, b] \to \boldsymbol{R}$ be two differentiable mappings with $g'(x) \neq 0$ on (a, b) and there exists a constant $M > 0$ so that :

$$\left| \frac{f'(x)}{g'(x)} \right| \leq M \text{ for all } x \in (a, b). \qquad \dots (4.3)$$

Then we have the inequality (4.2). The inequality is sharp.

PROOF : Use the Cauchy's mean value theorem, i.e., for every $x, y \in [a, b]$ with $x \neq y$, there exists a c between x and y so that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c)}{g'(c)}.$$

Consequently, for each $x, y \in [a, b]$ we have

$$\mid f(x) - f(y) \mid \leq M \mid g(x) - g(y) \mid$$

i.e., (4.1) holds. Applying Theorem 4.1, we get (4.3). ■

Remark 4.1 : Under the assumption of Corollary 4.2 we can choose

$$M = \sup_{x \in (a, b)} \left| \frac{f'(x)}{g'(x)} \right| = \left\| \frac{f'}{g'} \right\|_\infty,$$

assuming that the norm is finite.

Remark 4.2 : If f, g are as in the above theorem, then we have the inequality

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right|$$

$$\leq M \left[\frac{1}{b-a} \int_a^b g^2(x)dx - \left(\frac{1}{b-a} \int_a^b g(x)dx \right)^2 \right] \qquad \dots (4.4)$$

and the inequality is sharp.

2. If f, g are as in Corollary 4.2, then we have the inequality

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right|$$

$$\leq \left\| \frac{f'}{g'} \right\|_\infty \left[\frac{1}{b-a} \int_a^b g^2(x)dx - \left(\frac{1}{b-a} \int_a^b g(x)dx \right)^2 \right]$$

and the inequality is sharp.

5. The Case when Both Mappings are of Hölder Type

In this section we point out a Grüss' type inequality for mappings satisfying the condition of Hölder as follows :-

Theorem 5.1 — *Suppose that f is of r-Hölder type and g is of s-Hölder, i.e.,*

$$|f(x) - f(y)| \leq H_1 |x-y|^r \text{ and } |g(x) - g(y)| \leq H_2 |x-y|^s. \qquad \dots (5.1)$$

for all $x, y \in [a, b]$, *where* $H_1, H_2 > 0$ *and* $r, s \in (0, 1]$ *are fixed. Then we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right|$$

$$\leq \frac{H_1 H_2 (b-a)^{r+2}}{(r+s+1)(r+s+2)}. \qquad \dots (5.2)$$

Proof : By the assumption (5.1) we have

$$|(f(x) - f(y))(g(x) - g(y))| \leq H_1 H_2 |x-y|^{r+s}$$

for all $x, y \in [a, b]$.

Integrating on $[a, b]^2$ we get

$$\left| \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y))\, dxdy \right|$$

$$\leq \int_a^b \int_a^b |(f(x) - f(y))(g(x) - g(y))|\, dxdy \leq H_1 H_2 \int_a^b \int_a^b |x-y|^{r+s} dxdy.$$

Now, we observe that :

$$\int_a^b \int_a^b |x-y|^{r+s} dxdy = \int_a^b \left(\int_a^b |y-x|^{r+s} dy \right) dx$$

$$= \int_a^b \left(\int_a^x (x-y)^{r+s} dy + \int_x^b (y-x)^{r+s} dy \right) dx$$

$$= \int_a^b \left[\frac{(x-a)^{r+s+1} + (b-x)^{r+s+1}}{r+s+1} \right] dx$$

$$= \frac{2(b-a)^{r+s+1}}{(r+s+1)(r+s+2)}$$

and as

$$\frac{1}{2} \int_a^b \int_a^b (f(x)-f(y))(g(x)-g(y))\,dxdy$$

$$= (b-a) \int_a^b f(x)g(x)dx - \int_a^b f(x)dx \cdot \int_a^b g(x)dx$$

we get the desired inequality (5.2). ■

6. The Case When f' and g' Belong to Some L_p-Spaces

In this section we point out some inequalities of Grüss' type for differentiable mappings whose derivatives belong firstly to $L_\infty(a, b)$, then $L_p(a, b)$ $(p > 1)$ and finally to $L_1(a, b)$.

Theorem 6.1 — *Let $f, g : [a, b] \to \mathbf{R}$ be two differentiable mapings on (a, b) and $p : [a, b] \to [0, \infty)$ is integrable on [a, b]. If $f', g' \in L_\infty(a, b)$, then we have the inequality*

$$\left| \int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx \right|$$

$$\leq \frac{1}{2} \int_a^b \int_a^b p(x)p(y) \left| \int_x^y |f'(t)|\,dt \right| \left| \int_x^y |g'(z)|\,dz \right| dxdy$$

$$\leq \|f'\|_\infty \|g'\|_\infty \left[\int_a^b p(x)dx \int_a^b p(x)x^2dx - \left(\int_a^b p(x)xdx \right)^2 \right]. \qquad \text{... (6.1)}$$

Moreover, the inequality (6.1) is sharp.

PROOF : Let us observe that for any $x, y \in [a, b]$ we have

$$(f(x)-f(y))\,(g(x)-g(y)) = \int_x^y \int_x^y f'(t)g'(z)\,dtdz.$$

As $f', g' \in L_\infty(a, b)$, then we have

$$p(x)p(y)\,|\,(f(x)-f(y))\,(g(x)-g(y))\,|$$

$$\leq \left| \int_x^y |f'(t)|\,dt \right| \left| \int_x^y |g'(z)|\,dz \right| p(x)p(y) \leq \|f'\|_\infty \|g'\|_\infty (x-y)^2\, p(x)p(y)$$

for all $x, y \in [a, b]$.

By the properties of the modulus, we have

$$\left| \int_a^b \int_a^b p(x)p(y)(f(x)-f(y))\,(g(x)-g(y))\,dxdy \right|$$

$$\leq \int_a^b \int_a^b p(x)p(y) \left| \int_x^y |f'(t)|\,dt \right| \left| \int_x^y |g'(z)|\,dz \right| dxdy \quad \ldots (6.2)$$

$$\leq \|f'\|_\infty \|g'\|_\infty \int_a^b \int_a^b (x-y)^2\, p(x)p(y)\,dxdy,$$

from where we get the desired inequality (6.1).

To prove the sharpness of (6.1), let consider the mappings $f(x) = \alpha x + \beta$, $g(x) = \gamma x + \delta\,(\alpha, \gamma > 0, \beta, \delta \in \boldsymbol{R})$ on $[a, b]$. A simple calculation gives

$$\int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx$$

$$= \frac{1}{2} \int_a^b \int_a^b p(x)p(y) \left| \int_x^y |f'(t)|\,dt \right| \left| \int_x^y |g'(z)|\,dz \right| dxdy$$

$$= \|f'\|_\infty \|g'\|_\infty \left[\int_a^b p(x)dx \int_a^b p(x)x^2dx - \left(\int_a^b p(x)xdx \right)^2 \right]$$

$$= \frac{\alpha\gamma}{2} \int_a^b \int_a^b (x\;\;y)^2\, p(x)p(y)dxdy$$

which proves that we can have equality in all inequalities in (6.1). ■

The following corollary holds.

Corollary 6.2 — With the above assumptions on the mappings f, g we have :

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)\, dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right|$$

$$\leq \frac{1}{2} \int_a^b \int_a^b \left| \int_x^y |f'(t)|\, dt \right| \left| \int_x^y |g'(z)|\, dz \right| dxdy \leq \frac{\|f'\|_\infty \|g'\|_\infty}{12} (b-a)^2 \; . \qquad \ldots \tag{6.3}$$

The constants $\frac{1}{2}$ and $\frac{1}{12}$, respectively, are the best possible.

Remark 6.1 : We shall show that some time the estimation given by classical Grüss' inequality for the difference

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx - \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx$$

is better than the estimation (6.3) and some other time is the other way around.

Let $f, g : [0, 1] \to [0, \infty)$ given by $f(x) = x^p$, $g(x) = x^q$, $p, q > 1$. Then

$$\varphi = \inf_{x \in [0,1]} f(x) = 0, \quad \Phi = \sup_{x \in [0,1]} f(x) = 1;$$

$$\gamma = \inf_{x \in [0,1]} g(x) = 0, \quad \Gamma = \sup_{x \in [0,1]} g(x) = 1.$$

Also we have

$$f' = px^{p-1},\ g'(x) = qx^{q-1},\ x \in [0, 1]$$

and obviously $\|f'\|_\infty = p$, $\|g'\|_\infty = q$.

Now, we observe that

$$\frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma) = \frac{1}{4}$$

and

$$\frac{\|f'\|_\infty \|g'\|_\infty}{12} (b-a)^2 = \frac{pq}{12}.$$

Consequently, if $pq > 3$, then the bound provided by Grüss' inequality is better than the bound provided by (6.3). If $pq < 3$ $(p, q > 1)$ then (6.3) is better than (1.1).

Remark 6.2 : The inequality (6.3) is also a refinement of Cebyšev's inequality embodied in Corollary 2.2.

The following theorem also holds true.

Theorem 6.3 — *Let $f, g : [a, b] \to \mathbf{R}$ be two differentiable mappings on (a, b) and $p : [a, b] \to [0, \infty)$ is integrable on $[a, b]$. If $f \in L_\alpha(a, b), g' \in L_\beta(a, b)$ with $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then we have the inequality*

$$\left| \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx \right|$$

$$\leq \frac{1}{2} \left(\int_a^b \int_a^b p(x)p(y) \,|\, x - y \,|\, \left| \int_x^y |\, f'(t) \,|^\alpha dt \right| dxdy \right)^{\frac{1}{\alpha}}$$

$$\times \left(\int_a^b \int_a^b p(x)p(y) \,|\, x - y \,|\, \left| \int_x^y |\, g'(t) \,|^\alpha dt \right| dxdy \right)^{\frac{1}{\beta}}$$

$$\leq \frac{1}{2} \| f \|_\alpha \| g' \|_\beta \int_a^b \int_a^b |\, x - y \,|\, p(x)p(y)dxdy.$$

Note that, the first inequality in (6.4) is sharp.

PROOF : Using Hölder's inequality for double integrals, we have

$$\left| \int_x^y \int_x^y |\, f'(t)\, g'(z) \,|\, dtdz \right|$$

$$\leq \left| \int_x^y \int_x^y |\, f'(t) \,|^\alpha dtdz \right|^{\frac{1}{\alpha}} \left| \int_x^y \int_x^y |\, g'(z) \,|^\beta dtdz \right|^{\frac{1}{\beta}}$$

$$= |\, x - y \,|^{\frac{1}{\alpha}} \left| \int_x^y |\, f'(t) \,|^\alpha dt \right|^{\frac{1}{\alpha}} |\, x - y \,|^{\frac{1}{\beta}} \left| \int_x^y |\, g'(z) \,|^\alpha dz \right|^{\frac{1}{\beta}}$$

$$= |\, x - y \,| \left| \int_x^y |\, f'(t) \,|^\alpha dt \right|^{\frac{1}{\alpha}} \left| \int_x^y |\, g'(t) \,|^\beta dt \right|^{\frac{1}{\beta}}.$$

Now, as in the proof of Theorem 6.1, we have :

$$\left| \int_a^b \int_a^b p(x)p(y)\, (f(x) - f(y))\, (g(x) - g(y))\, dxdy \right|$$

$$\leq \int_a^b \int_a^b p(x)p(y) \left| \int_x^y \int_x^y |f'(t)\, g'(z)|\, dt dz \right| dxdy$$

$$\leq \int_a^b \int_a^b p(x)p(y)\, |x-y| \left| \int_x^y |f'(t)|^{\alpha} dt \right|^{\frac{1}{\alpha}} \left| \int_x^y |g'(z)|^{\beta} dz \right|^{\frac{1}{\beta}} dxdy.$$

Using again Hölder's inequality for double integrals, we have

$$\int_a^b \int_a^b p(x)p(y)\, |x-y| \left| \int_x^y |f'(t)|^{\alpha} dt \right|^{\frac{1}{\alpha}} \left| \int_x^y |g'(z)|^{\beta} dz \right|^{\frac{1}{\beta}} dxdy$$

$$\leq \left(\int_a^b \int_a^b p(x)p(y)\, |x-y| \left| \int_x^y |f'(t)|^{\alpha} dt \right| dxdy \right)^{\frac{1}{\alpha}}$$

$$\times \left(\int_a^b \int_a^b p(x)p(y)\, |x-y| \left| \int_x^y |g'(z)|^{\beta} dz \right| dxdy \right)^{\frac{1}{\beta}} \quad \ldots (6.5)$$

and, as

$$\int_a^b \int_a^b p(x)p(y)\, (f(x)-f(y))\, (g(x)-g(y))\, dxdy$$

$$= 2\left[\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \right] \quad \ldots (6.6)$$

the inequality (6.5) and (6.6) provide the first inequality in (6.4).

Now, let us observe that

$$\left| \int_x^y |f'(t)|^{\alpha} dt \right| \leq \| f' \|_{\alpha}^{\alpha}, \quad \left| \int_x^y |g'(z)|^{\beta} dz \right| \leq \| g' \|_{\beta}^{\beta}$$

for all $x, y \in [a, b]$, and then

$$\left(\int_a^b \int_a^b p(x)p(y)\, |x-y| \left| \int_x^y |f'(t)|^{\alpha} dt \right| dxdy \right)^{\frac{1}{\alpha}}$$

$$\times \left(\int_a^b \int_a^b p(x)p(y) \mid x-y \mid \left| \int_x^y \mid g'(z) \mid^{\beta} dt \right| dxdy \right)^{\frac{1}{\beta}}$$

$$\leq \|f'\|_{\alpha} \left(\int_a^b \int_a^b p(x)p(y) \mid x-y \mid dxdy \right)^{\frac{1}{\alpha}} \times \| g' \|_{\beta} \left(\int_a^b \int_a^b p(x)p(y) \mid x-y \mid dxdy \right)^{\frac{1}{\beta}}$$

$$= \|f'\|_{\alpha} \| g' \|_{\beta} \int_a^b \int_a^b p(x)p(y) \mid x-y \mid dxdy$$

and the second inequality in (6.4) is also proved.

For the sharpness of the first inequality in (6.4), let us consider the mappings $f, g : [a, b] \to \boldsymbol{R}, f(x) = mx + n$, $g(x) = sx + z$ with $m,\ t > 0$. Then, obviously

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx$$

$$= \frac{1}{2} ms \int_a^b \int_a^b p(x)p(y)\,(x-y)^2\,dxdy$$

and

$$\left| \int_x^y \mid f'(t) \mid^{\alpha} dt \right| = m^{\alpha} \mid x-y \mid, \left| \int_x^y \mid g'(z) \mid^{\beta} dz \right| = s^{\beta} \mid x-y \mid$$

then

$$\left(\int_a^b \int_a^b p(x)p(y) \mid x-y \mid \left| \int_x^y \mid f'(t) \mid^{\alpha} dt \right| dxdy \right)^{\frac{1}{\alpha}}$$

$$\times \left(\int_a^b \int_a^b p(x)p(y) \mid x-y \mid \left| \int_x^y \mid g'(z) \mid^{\beta} dz \right| dxdy \right)^{\frac{1}{\beta}}$$

$$= ms \left(\int_a^b \int_a^b p(x)p(y) \mid x-y \mid^2 dxdy \right)^{\frac{1}{\alpha}} \times \left(\int_a^b \int_a^b p(x)p(y) \mid x-y \mid^2 dxdy \right)^{\frac{1}{\beta}}$$

$$= ms \int_a^b \int_a^b p(x)p(y)(x-y)^2\, dxdy$$

and the equality is realized in the first inequality in (6.4). ■

The following corollary holds.

Corollary 6.4 — Let f, g be as above. Then we have the inequality

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right|$$

$$\leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b p(x)p(y)\,|x-y| \sup_{t\in[x,y]} |f'(t)| \left| \int_x^y |g'(z)|\,dz \right| dxdy \quad \text{... (6.9)}$$

$$\leq \frac{1}{6} \|f'\|_\infty \|g'\|_1 (b-a).$$

The first inequality in (6.9) is sharp.

Remark 6.3 : We note that some time the upper bound provided by (6.4) is better than the upper bound given by (6.8) and other time, the other way around.

Indeed, choosing $f, g : [0, 1] \to \boldsymbol{R}, f(x) = x^p, g(x) = x^q$ $(p, q > 1)$ we have

$$f'(x) = px^{p-1}, g'(x) = qx^{q-1}, \|f'\|_\infty = p, \|g'\|_1 = 1,$$

$$\|f'\|_\alpha = \frac{p}{[\alpha(p-1)+1]^{\frac{1}{\alpha}}}$$

and

$$\|g'\|_\alpha = \frac{q}{[\beta(q-1)+1]^{\frac{1}{\beta}}}$$

where $\alpha, \beta > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Also, let

$$A := \frac{1}{6} \|f'\|_\infty \|g'\|_1 (b-a) = \frac{p}{6}$$

and

$$B := \frac{1}{6} \|f'\|_\alpha \|g'\|_\beta (b-a) = \frac{pq}{6[\alpha(p-1+1)^{\frac{1}{\alpha}} [\beta(q-1)+1]^{\frac{1}{\beta}}}.$$

If we choose $\alpha = \beta = 2$, we get

$$\frac{A}{B} = \frac{[(2p+1)(2q+1)]^{\frac{1}{2}}}{q}$$

which can be greater less than 1 for different values of p, $q > 1$.

REFERENCE

1. D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, *Classical and new Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.