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INFINITE BOUNDARY VALUE PROBLEMS FOR NONLINEAR IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS IN A BANACH SPACE*

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This article is devoted to the study of infinite boundary value problems for nonlinear first order impulsive integro-differential equations in a Banach space by means of the upper and lower solution method and the monotone iterative technique.

KeyWords : **Banach Space; Impulsive Integro-Differential Equation; Infinite Boundary Value Problem; Comparison Result; Regular Cone**

1. INTRODUCTION AND PRELIMINARIES

In recent years the theory of impulsive differential equations has become an important area of research, many mathematicians have investigated the boundary value problems for such equations in detail (see Lakshmikantham, Bainov and Simenov[1]). However, all of their work is restricted to finite intervals. In the present article, after a new comparison result is established, the author will apply the upper and lower solution method and the monotone iterative technique to the boundary value problems for nonlinear first order impulsive integro-differential equations in Banach space E defined on an infinite interval, which we name infinite boundary value problems (IFBVP for brevity).

Let E be partially ordered by its cone K, i.e., for $u, v \in E, u \leq v$ iff $v - u \in K$. K is called normal if there exists a positive constant N such that $\theta \leq u \leq v$ implies $\| u \| \leq N \| v \|$, and K is called regular if $u_1 \leq u_2 \leq \ldots \leq u_n \leq \ldots \leq v$ implies $u_n \to u$ as $n \to +\infty$ for some $u \in E$. It is well known that the regularity of K implies the normality of K (see Guo and Lakshmikantham[2]). We shall consider the IFBVP :

$$\begin{cases} u' = f(t, u, Tu, Su), t \in J, t \neq t_k, \\ \Delta u|_{t = t_k} = I_k(u(t_k)), k = 1, 2, \ldots, \\ u(0) = u(\infty), \end{cases} \quad \ldots (1.1)$$

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where, $J=[0,+\infty), f\in C[J\times E\times E\times E, E], 0<t_1<t_2<\ldots<t_k<\ldots,$ and $t_k\to+\infty$ as $k\to+\infty$, $I_k\in C[E,E], k = 1,2,\ldots, u(\infty) = \lim_{t\to+\infty} u(t), (Tu)(t)=\int_0^t k(t,s)u(s)ds, (Su)(t)=\int_0^\infty h(t,s)u(s)ds,$ $k\in C[D,R_+], h = C[J\times J, R_+], D = \{(t,s)\in J\times J: t\geq s\}$. $\Delta u|_{t=t_k}=u(t_k^+)-u(t_k^-)$ denotes the jump of $u(t)$ at $t=t_k$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t=t_k$, respectively. Set $PC[J,E]=\{x:J\to E: x(t)$ is continuous at $t\neq t_k$, left continuous at $t=t_k$ and each $x(t_k^+)$ exists, for $k = 1, 2, \ldots\}$, $BPC[J,E]=\left\{x\in PC[J,E]: \sup_{t\in J}\|x(t)\|<+\infty\right\}$, $TPC[J,E] = x\in BPC[J,E]$: $\lim_{t\to+\infty} x(t)=x(\infty)$ exists. Evidently, $BPC[J,E]$ endowed with the norm $\|x\|_B=\sup_{t\in J}\|x\|$ is a Banach space and $TPC[J,E]\subset BPC[J,E]$. Denote $J' = J\setminus\{t_1,t_2,\ldots,t_k,\ldots\}$, $J_0=[0,t_1], J_k=(t_k,t_{k+1}]$, for $k = 1, 2, \ldots$. By a solution of IFBVP (1.1) we mean $x\in TPC[J,E]\cap C^1[J',E]$ satisfying (1.1).

2. Comparison Result and Other Lemmas

We first consider the linear IFBVP:

$$\begin{cases} u'=f(t,\eta,T\eta,S\eta)-M(t)(u-\eta)-N(t)(Tu-T\eta)-N_1(t)(Su-S\eta), t\in J, t\neq t_k \\ \Delta u|_{t=t_k}=I_k(\eta(t_k))-L_k[u(t_k)-\eta(t_k)], k=1,2,.., \\ u(0)=u(\infty) \end{cases} \qquad \ldots (2.1)$$

where, $\eta\in BPC[J,E], M, N, N_1\in C[J,R_+]\cap L^1(J)$.

We shall now establish the crucial comparison theorem of the paper.

Lemma 1 — (comparison result) Let $\int_0^t k(t,s)\,ds, \int_0^\infty h(t,s)ds\in L^1(J)$ $p\in TPC[J,E]\cap C[J',E]$. Assume that there exist $M,N,N_1\in C[J,R_+]\cap L^1(J)$, with $M\not\equiv 0$, and constants $L_k\geq 0, k= 1, 2, \ldots$, such that

$$\begin{cases} p'\leq -M(t)p(t)-N(t)(Tp)(t)-N_1(t)(Sp)(t), t\in J, t\neq t_k, \\ \Delta p|_{t=t_k}\leq -L_kp(t_k), k=1,2,\ldots, \\ p(0)\leq p(\infty) \end{cases} \qquad \ldots (2.2)$$

and

$$e^{\int_0^\infty M(s)ds}$$

$$\left\{ e^{\int_0^\infty M(s)ds} \left[\int_0^\infty N(t)dt \int_0^t k(t,s)ds + \int_0^\infty N_1(t)dt \int_0^\infty h(t,s)ds \right] + \sum_{k=1}^\infty L_k \right\} \le 1. \quad \ldots (2.3)$$

Then $p(t) \le \theta$, for $t \in J$.

PROOF : For any given $g \in K^*$ (the dual cone of K), let $w(t) = g(p(t))$, then $w \in TPC[J, R] \cap C^1 [J', R]$, $w'(t) = g(p'(t))$, $(Tw)(t) = g((Tp)(t))$, $(Sw)(t) = g((Sp)(t))$, and so (2.2) yields that

$$\begin{cases} w' \le -M(t)w(t) - N(t)(Tw)(t) - N_1(t)(Sw)(t), \; t \in J, \, t \ne t_k, \\ \Delta w|_{t=t_k} \le -L_k w(t_k), \; k = 1, 2, \ldots, \\ w(0) \le w(\infty). \end{cases} \quad \ldots (2.4)$$

Set $v(t) = w(t)\, e^{\int_0^t M(s)ds}$, then (2.4) gives that

$$\begin{cases} v' \le -N(t) \int_0^t k^*(t,s)v(s)ds - N_1(t) \int_0^\infty h^*(t,s)v(s)ds, \; t \in J, \, t \ne t_k, \\ \Delta v|_{t=t_k} \le -L_k v(t_k), \; k = 1, 2, \ldots, \\ v(0) \le v(\infty)\, e^{-\int_0^\infty M(s)ds}, \end{cases} \quad \ldots (2.5)$$

where $k^*(t,s) = k(t,s)\, e^{\int_s^t M(r)dr}$, $h^*(t,s) = h(t,s) e^{\int_s^t M(r)dr}$. We claim that

$$v(t) \le 0, \text{ for } t \in J. \quad \ldots (2.6)$$

Suppose the contrary to (2.6). To arrive at a contradiction, we shall next investigate either (a) : $v(t) \ge 0$ for $t \in J$, and there exists $t_1^* \in J$ such that $v(t_1^*) > 0$, or (b) : there exist $t_1^*, t_2^* \in J$ such that $v(t_1^*) > 0$, $v(t_2^*) < 0$.

In case (a), then (2.5) yields that $v'(t) \le 0$ for $t \in J, t \ne t_k$. On the other hand, $v(t_k^+) = v(t_k) + \Delta v|_{t=t_k} \le (1 - L_k)\, v(t_k) \le v(t_k)$ (since (2.3) implies $L_k \le 1$). Thus $v(t)$ is nonincreasing and thus $v(0) \ge v(t_1^*) > 0$, $v(0) \ge v(\infty)$. Therefore again by (2.5), we see that $v(0) \ge v(\infty) \ge v(0)\, e^{\int_0^\infty M(s)ds} > v(0)$, a contradiction.

In case (b), let $\inf_{t \in J} v(t) = -d$, then $d > 0$, and then we have either that (b_1): there exists some J_i such that either $v(t_0^*) = -d$ for some $t_0^* \in J_i$, or $v(t_i^+) = -d$, or that (b_2): $v(\infty) = -d$.

In subcase (b_1), we only discuss the case of $v(t_0^*) = -d$ for $t_0^* \in J_i$, since the case of $v(t_i^+) = -d$ is similar. If $v(t_0^*) = -d$ for $t_0^* \in J_i$, then we show from (2.5) that for $t \in J, t \ne t_k$,

$$v'(t) \le d\left[N(t) \int_0^t e^{\int_s^t M(r)dr} k(t,s)ds + N_1(t) \int_0^\infty e^{\int_s^t M(r)dr} h(t,s)ds \right]$$

$$\le de^{\int_0^\infty M(s)ds} \left[N(t) \int_0^t k(t,s)ds + N_1(t) \int_0^\infty h(t,s)ds \right]. \qquad \text{... (2.7)}$$

For any integer $m \ge i$, the calculus fundamental principle implies that

$$v(t_{m+1}) - v(t_m^+) = \int_{t_m}^{t_{m+1}} v'(t)dt,$$

$$v(t_m) - v(t_{m-1}^+) = \int_{t_{m-1}}^{t_m} v'(t)dt,$$

....

$$v(t_{i+2}) - v(t_{i+1}^+) = \int_{t_{i+1}}^{t_{i+2}} v'(t)dt,$$

$$v(t_{i+1}) - v(t_0^*) = \int_{t_0^*}^{t_{i+1}} v'(t)dt.$$

Hence, the above $(m - i + 1)$ formulae just obtained, together with (2.7), yield that

$$v(t_{m+1}) - v(t_0^*) \le \int_{t_0^*}^{t_{m+1}} v'(t)dt + \sum_{k=i+1}^{m} \Delta v|_{t=t_k}$$

$$\le de^{\int_0^\infty M(s)ds} \int_{t_0^*}^{t_{m+1}} \left[N(t) \int_0^t k(t,s)ds + N_1(t) \int_0^\infty h(t,s)ds \right] dt + d \sum_{k=i+1}^{m} L_k. \qquad \text{... (2.8)}$$

By passing to limits in (2.8) as $m \to +\infty$, we show that

$$v(\infty) + d = v(\infty) - v(t_0^*)$$

$$\le de^{\int_0^\infty M(s)ds} \int_{t_0^*}^{\infty} \left[N(t) \int_0^t k(t,s)ds + N_1(t) \int_0^\infty h(t,s)ds \right] dt + d \sum_{k=i+1}^{\infty} L_k$$

$$\leq de^{\int_0^\infty M(s)ds}\left[\int_0^\infty N(t)dt\int_0^t k(t,s)ds+\int_0^\infty N_1(t)dt\int_0^\infty h(t,s)ds\right]+d\sum_{k=1}^\infty L_k \quad \ldots (2.9)$$

We claim $v(\infty)\leq 0$. Suppose the contrary, then we obtain from (2.9) that

$$e^{\int_0^\infty M(s)ds}\left[\int_0^\infty N(t)dt\int_0^t k(t,s)ds+\int_0^\infty N_1(t)dt\int_0^\infty h(t,s)ds\right]+\sum_{k=1}^\infty L_k>1,$$

which contradicts (2.3). Hence $v(\infty)\leq 0$, and $v(0)\leq v(\infty)\,e^{-\int_0^\infty M(s)ds}\leq 0$. Therefore, $0<t_1^*<+\infty$. Without loss of generality, we assume that $t_1^*\in J_j$ for some j.

If $t_0^*<t_1^*$, then $i\leq j$. Employing an argument similar to the establishment of (2.8), we see that

$$d<v(t_1^*)-v(t_0^*)$$

$$\leq de^{\int_0^\infty M(s)ds}\int_{t_0^*}^{t_1^*}\left[N(t)\int_0^t k(t,s)ds+N_1(t)\int_0^\infty h(t,s)ds\right]dt+\sum_{k=i+1}^{j} L_k$$

$$\leq de^{\int_0^\infty M(s)ds}\left[\int_0^\infty N(t)dt\int_0^t k(t,s)ds+\int_0^\infty N_1(t)dt\int_0^\infty h(t,s)ds\right]+d\sum_{k=1}^\infty L_k,$$

hence,

$$e^{\int_0^\infty M(s)ds}\left[\int_0^\infty N(t)dt\int_0^t k(t,s)ds+\int_0^\infty N_1(t)dt\int_0^\infty h(t,s)ds\right]+\sum_{k=1}^\infty L_k>1$$

which also contradicts (2.3).

If $t_0^*>t_1^*$, then $i\geq j$. Using almost the same arguments as before, we obtain that

$$v(t_1^*)-v(0)$$

$$\leq de^{\int_0^\infty M(s)ds}\int_0^{t_1^*}\left[N(t)\int_0^t k(t,s)ds+N_1(t)\int_0^\infty h(t,s)ds\right]dt+d\sum_{k=1}^{j} L_k=:\sum\nolimits_1$$

$$\ldots (2.10)$$

$$v(\infty)-v(t_0^*)$$

$$\leq de^{\int_0^\infty M(s)ds} \int_{t_0^*}^{\infty_1} \left[N(t) \int_0^t k(t,s)ds + N_1(t) \int_0^\infty h(t,s)ds \right] dt + d \sum_{k=i+1}^{\infty} L_k =: \sum\nolimits_2$$

... (2.11)

Multiplying (2.10) by $e^{\int_0^\infty M(s)ds}$ and adding (2.11), we show from (2.5) that

$$d < e^{\int_0^\infty M(s)ds} v(t_1^*) - e^{\int_0^\infty M(s)ds} v(0) + v(\infty) - v(t_0^*)$$

$$\leq e^{\int_0^\infty M(s)ds} \sum\nolimits_1 + \sum\nolimits_2 = \left(e^{\int_0^\infty M(s)ds} - 1 \right) \sum\nolimits_1 + \left(\sum\nolimits_1 + \sum\nolimits_2 \right)$$

$$\leq de^{\int_0^\infty M(s)ds}$$

$$\left\{ e^{\int_0^\infty M(s)ds} \left[\int_0^\infty N(t)dt \int_0^t k(t,s)ds + \int_0^\infty N_1(t)dt \int_0^\infty h(t,s)ds \right] + \sum_{k=1}^{\infty} L_k \right\},$$

which immediately yields a contradiction to (2.3).

In subcase (b_2), then it follows from (2.5) that

$$v(0) \leq v(\infty)\, e^{-\int_0^\infty M(s)ds} = -de^{-\int_0^\infty M(s)ds},$$

which, together with (2.10), yields that

$$de^{-\int_0^\infty M(s)ds} < v(t_1^*) - v(0)$$

$$\leq de^{\int_0^\infty M(s)ds} \int_0^{t_1^*} \left[N(t) \int_0^t k(t,s)ds + N_1(t) \int_0^\infty h(t,s)ds \right] dt + d \sum_{k=1}^{j} L_k$$

$$\leq de^{\int_0^\infty M(s)ds} \left[\int_0^\infty N(t)dt \int_0^t k(t,s)ds + \int_0^\infty N_1(t)dt \int_0^\infty h(t,s)ds \right] + d \sum_{k=1}^{\infty} L_k .$$

From this we easily obtain a contradiction to (2.3).

Hence, in view of the above arguments, we conclude that $v(t) \leq 0$ for $t \in J$, and furthermore

$w(t) = v(t)\, e^{-\int_0^t M(s)ds} \leq 0$. Since $g \in K^*$ is arbitrarily given, therefore $p(t) \leq 0$ for $t \in J$. The proof is complete.

Lemma 2 — Let M, N, N_1, $\int_0^t k(t, s)ds$, and $\int_0^\infty h(t, s)ds$ be as in Lemma 1, assume also that

(H_1) there exist $\alpha, \beta, \gamma \delta \in C[J, R_+] \cap L^1(J)$ such that

$$\| f(t, u, v, w) \| \leq \alpha(t) \| u \| + \beta(t) \| v \| + \gamma(t) \| w \| + \delta(t), \text{ for } t \in J, u, v, w \in E;$$

(H_2) there exist $L_k \geq 0, k = 1, 2, \ldots$, such that $\sum_{k=1}^{\infty} L_k$ is convergent and

$$\| I_k(u) \| \leq L_k \| u \|, \text{ for } u \in E.$$

Then for any $\eta \in BPC[J, E]$ fixed, $u \in TPC[J, E] \cap C^1[J', E]$ is a solution of IFBVP (2.1) if and only if $u \in BPC[J, E]$ is a solution of the following impulsive integral equation (IIE):

$$u(t) = e^{-\int_0^t M(s)ds} \left[\left(e^{\int_0^\infty M(s)ds} - 1 \right)^{-1} \left[\int_0^\infty e^{\int_0^s M(r)dr} (f(s, \eta, T\eta, S\eta) + M(s)\eta(s) \right.\right.$$

$$\left. - N(s)(Tu - T\eta) - N_1(s)(Su - S\eta))\, ds + \sum_{k=1}^{\infty} e^{\int_0^{t_k} M(s)ds} (I_k(\eta(t_k)) - L_k(u(t_k) - \eta(t_k))) \right]$$

$$\left. + \int_0^t e^{\int_0^s M(r)dr} (f(s, \eta, T\eta, S\eta) + M(s)\eta(s) - N(s)(Tu - T\eta) - N_1(s)(Su - S\eta))ds \right]$$

$$+ \sum_{0 < t_k < t} e^{-\int_{t_k}^t M(s)ds} (I_k(\eta(t_k)) - L_k(u(t_k) - \eta(t_k))). \qquad \ldots (2.12)$$

PROOF : To prove 'if'. If $u \in BPC[J, E]$ is a solution of IIE (2,12) then (H_1) and (H_2) imply that

$$\| f(s, \eta, T\eta, S\eta) + M(s)\eta(s) - N(s (Tu - T\eta) - N_1(s)(Su - S\eta) \|$$

$$\leq \left[M(s) + \alpha(s) + (\beta(s) + N(s)) \int_0^s k(s, r)dr + (\gamma(s) + N_1(s)) \int_0^\infty h(s, r)dr \right] \| \eta \|_B$$

$$+\left[N(s)\int_0^s k(r,s)dr+N_1(r)\int_0^\infty h(s,r)dr\right]\|u\|_B+\delta(s)=:Q(s),\ \text{for } s\in J$$

$$\|I_k(\eta(t_k))-L_k(u(t_k)-\eta(t_k))\|$$

$$\leq(\bar{L}_k+L_k)\|\eta\|_B+L_k\|u\|_B=q_k,$$

and $Q(s)\in L^1(J)$, $\sum_{k=1}^\infty q_k$ is convergent, thus the right hand side of (2.12) is well defined. Moreover, we show by direct computation that $u\in TPC[J,E]\cap C^1[J',E]$ is a solution of IFBVP (2.1). To prove only if, provided $U\in TPC[J,E]\wedge C'[J',E]$ is a solution of IFBVP (2.1), then by the proof of 'if', we only need to show the uniqueness of solution u. Let u_1, u_2 be any two solutions of IFBVP (2.1), and $p=u_1-u_2$, then a direct check yields that

$$\begin{cases} p'(t)=-M(t)p(t)-N(t)(Tp)(t)-N_1(t)(Sp)(t),\ t\in J,\ t\neq t_k,\\ \Delta p|_{t=t_k}=-L_kp(t_k),\ k=1,2,\ldots,\\ p(0)=p(\infty).\end{cases}$$

Hence, Lemma 1 implies that $p(t)\leq\theta$ for $t\in J$, i.e. $u_1\leq u_2$; analogously, $u_2\leq u_1$. Therefore, $u_1=u_2$. We complete the proof.

We define an operator A as follows : for any $\eta\in BPC[J,E]$, denote by $(A\eta)(t)$ the right hand side of (2.12).

An application of Lemma 2 readily shows

Lemma 3 — Assume that all the conditions of Lemma 2 are satisfied, then $u\in TPC[J,E]\cap C^1[J',E]$ is a solution of IFBVP (1.1) if and only if $u\in BPC[J,E]$ is a fixed point of A.

Lemma 4 — Let (H_1) and (H_2) hold. Assume further that

$$c=\left(e^{\int_0^\infty M(s)ds}-1)\right)^{-1}\left\{\left(2e^{\int_0^\infty M(s)ds}-1\right.\right.$$

$$\left[\int_0^\infty N(t)dt\int_0^t k(t,s)ds+\int_0^\infty N_1(t)dt\int_0^\infty h(t,s)ds\right]$$

$$\left.+\sum_{k=1}^\infty L_k\left[e^{\int_0^\infty M(s)ds}+e^{\int_0^{t_k} M(s)ds}-1\right]\right\}<1. \qquad \ldots (2.13)$$

Then for any $\eta \in BPC[J, E]$, IIE (2.12) possesses a unique solution $u \in BPC[J, E]$. Moreover, $u = A\eta$.

PROOF : For any $\eta \in BPC[J, E]$ fixed, we define another operator G by $(Gu)(t)$ being the right hand side of (2.12).

By virtue of $(H_1)(H_2)$, it is obvious that G maps $BPC[J, E]$ into $BPC[J, E]$. Then for any $u_1, u_2 \in BPC[J, E], t \in J$,

$$\| (Gu_1)(t) - (Gu_2)(t) \| \cdot$$

$$\Bigg\| e^{-\int_0^t M(s)ds} \Bigg\{ \left(e^{\int_0^\infty M(s)ds} - 1 \right)^{-1}$$

$$\Bigg[\int_0^\infty e^{\int_0^s M(r)dr} (-N(s)(Tu_1 - Tu_2) - N_1(s)(Su_1 - Su_2))ds$$

$$+ \sum_{k=1}^\infty e^{\int_0^{t_k} M(s)ds} (-L_k(u_1(t_k) - u_2(t_k))) \Bigg] + \int_0^t e^{\int_0^s M(r)dr} (-N(s)(Tu_1 - Tu_2) - N_1(s)$$

$$(Su_1 - Su_2))\, ds \Bigg\} + \sum_{0<t_k<t} e^{-\int_{t_k}^t M(s)ds} (-L_k(u_1(t_k) - u_2(t_k))) \Bigg\|$$

$$\le \left(e^{\int_0^\infty M(s)ds} - 1 \right)^{-1} \Bigg[\int_0^\infty e^{\int_0^s M(r)dr} (N(s) \| (Tu_1)(s) - (Tu_2)(s) \|$$

$$+ N_1(s) \| (Su_1)(s) - (Su_2)(s) \|)ds$$

$$+ \sum_{k=1}^\infty e^{\int_0^{t_k} M(s)ds} L_k \| u_1(t_k) - u_2(t_k) \| \Bigg] + \int_0^t e^{-\int_s^t M(r)dr} (N(s) \| (Tu_1)(s) - (Tu_2)(s) \|$$

$$+ N_1(s) \| (Su_1)(s) - (Su_2)(s) \|)\, ds + \sum_{0<t_k<t} L_k e^{-\int_{t_k}^t M(s)ds} \| u_1(t_k - u_2(t_k) \|$$

$$\le \left(e^{\int_0^\infty M(s)ds} - 1 \right)^{-1} \Bigg[\left(2e^{\int_0^\infty M(s)ds} - 1 \right) \left(\int_0^\infty N(s)ds \cdot \int_0^s k(s, r)dr \right.$$

$$\left. + \int_0^\infty N_1(s)ds \int_0^\infty h(s, r)dr \right) + \sum_{k=1}^\infty L_k \left(e^{\int_0^\infty M(s)ds} + e^{\int_0^{t_k} M(s)ds} - 1 \right) \Bigg] \| u_1 - u_2 \|_B.$$

Thus $\| Gu_1 - Gu_2 \|_B \le c \| u_1 - u_2 \|_B$. Hence by (2.13), Banach's fixed point theorem implies that G has a unique fixed point $u \in BPC[J, E]$, i.e. a unique solution of IIE (2.12). On the other hand, by IIE (2.12) and the definitions of A, G, we have $A\eta = GA\eta$. Therefore, we conclude that $u = A\eta$. This proof is complete.

Remark 1 : Under the conditions of Lemma 4, A also maps $BPC[J, E]$ into $BPC[J, E]$.

3. Main Theorem

For our purpose, we also need the assumptions :

(H_3) Let $\int_0^t k(t, s)ds$, $\int_0^\infty h(t, s)ds \in L^1(J)$, and

$$\lim_{t' \to t} \int_0^\infty | h(t', s) - h(t, s) | ds \to 0 \text{ for } t \in J;$$

(H_4) Let $v_0, w_0 \in TPC[J, E] \cap C'[J', E]$ be the upper and lower solutions of IFBVP (1.1), respectively, i.e.

$$\begin{cases} v_0' \ge f(t, v_0, Tv_0, Sv_0), \; t \in J, \; t \ne t_k, \\ \Delta v_0|_{t = t_k} \ge I_k(v_0(t_k)), \; k = 1, 2, \ldots, \\ v_0(0) \ge v_0(\infty), \end{cases} \quad \ldots (3.1)$$

and

$$\begin{cases} w_0' \le f(t, w_0, Tw_0, Sw_0), \; t \in J, \; t \ne t_k, \\ \Delta w_0|_{t = t_k} \le I_k(w_0(t_k)), \; k = 1, 2, \ldots, \\ w_0(0) \le w_0(\infty), \end{cases} \quad \ldots (3.2)$$

Moreover,

$$w_0(t) \le v_0(t), \text{ for } t \in J; \quad \ldots (3.3)$$

(H_5) there exist $M, N, N_1 \in C[J, R_+] \cap L^1(J)$ with $M \not\equiv 0$, such that

$$f(t, u, v, w) - f(t, x, y, z) \ge -M(t)(u - x) - N(t)(v - y) - N_1(t)(w - z),$$

for $t \in J$, $w_0(t) \le x \le u \le v_0(t)$, $(Tw_0)(t) \le y \le v \le (Tv_0)(t)$, $(Sw_0)(t) \le z \le w \le (Sv_0)(t)$;

(H_6) there exist $L_k \ge 0$ $(k = 1, 2, \ldots)$ such that

$$I_k(u) - I_k(x) \ge -L_k(u - x), \text{ for, } w_0(t_k) \le x \le u \le v_0(t_k), \; k = 1, 2, \ldots .$$

Set the ordered interval $[w_0, v_0] = \{x \in BPC[J, E] : w_0(t) \le x(t) \le v_0(t), t \in J\}$.

Theorem 1 — *Let K be a regular cone in E, and the assumptions $(H_1)-(H_6)$ hold. Assume also that the inequalities (2.3), (2.13) are satisfied. Then IFBVP (1.1) possesses in $[w_0, v_0]$ the minimal and maximal solutions $w_*, v^* \in TPC[J, E] \cap C^1[J', E]$, respectively. Moreover, the monotone sequences $\{w_n(t)\}$ and $\{v_n(t)\}$ converge on J (uniformly on each J_k, $k = 1, 2, \ldots$) to $w_*(t), v^*(t)$ for $t \in J$, respectively. where,*

$$w_n(t) = Aw_{n-1}(t),\ v_n(t) = Av_{n-1}(t), \text{ for } t \in J, n = 1, 2, \ldots . \qquad \ldots (3.4)$$

PROOF : For any $\eta \in [w_0, v_0]$, Lemma 4 and Lemma 2 imply that $A\eta \in TPC[J, E] \cap C^1[J', E]$ is a unique solution of IFBVP (2.1). We shall show that : (*i*) $w_0 \le Aw_0$, $Av_0 \le v_0$; (*ii*) A is nondecreasing in $[w_0, v_0]$.

To prove (i) : Let $p = w_0 - w_1$, then in view of (3.1), Lemma 2 implies that

$$p'(t) \le -M(t)p(t) - N(t)(Tp)(t) - N_1(t)(Sp)(t),\ t \in J, t \ne t_k,$$

$$\Delta p|_{t = t_k} \le -L_k\, p(t_k),\ k = 1, 2, \ldots,$$

$$p(0) \le p(\infty),$$

which implies by Lemma 1 that $p(t) \le \theta$, for $t \in J$, i.e., $w_0 \le Aw_0$. Analogously, we show from (3.2) that $Av_0 \le v_0$.

To prove (ii) : Let $\eta_1, \eta_2 \in [w_0, v_0]$ with $\eta_1 \le \eta_2$, and $p = u_1 - u_2$, where $u_1 = A\eta_1$, $u_2 = A\eta_2$. Then by (H_5), (H_6), Lemma 2 yields that

$$p'(t) \le -M(t)p(t) - N(t)(Tp)(t) - N_1(t)(Sp)(t),\ t \in J, t \ne t_k,$$

$$\Delta p|_{t = t_k} \le -L_k\, p(t_k),\ k = 1, 2, \ldots,$$

$$p(0) = p(\infty),$$

Thus by Lemma 1, we show that $p(t) \le \theta$ for $t \in J$, i.e., $A\eta_1 \le A\eta_2$.

Hence by (*i*), (*ii*) just proved, it follows from (3.3), (3.4) that

$$w_0(t) \le w_1(t) \le \cdots \le w_n(t) \le \cdots \le v_n(t) \le \cdots \le v_1(t) \le v_0(t), \text{ for } t \in J. \qquad \ldots (3.5)$$

Therefore, by the regularity of K, it follows from (3.5) that

$$\lim_{n \to +\infty} w_n(t) = w_*(t), \text{ for } t \in J \qquad \ldots (3.6)$$

and by the normality of K, there exists a constant $L > 0$ such that

$$\| w_n \|_B \le L,\ n = 1, 2, \cdots . \qquad \ldots (3.7)$$

Hence, by (H_1), (H_2), we see that

$$\| f(s, w_{n-1}, Tw_{n-1}, Sw_{n-1}) + M(s)w_{n-1}(s) - N(s)(Tw_n - Tw_{n-1}) - N_1(s)(Sw_n - Sw_{n-1}) \|$$

$$\leq L(M(s) + \alpha(s) + (\beta(s) + 2N(s)) \int_0^s k(s, r)dr + (\gamma(s) + 2N_1(s) \int_0^\infty h(s, r)dr] + \delta(s)$$

$$=: R(s), \text{ for } s \in J, \quad \dots (3.8)$$

and

$$d\| I_k(w_{n-1}(t_k)) - L_k(w_n(t_k) - w_{n-1}(t_k)) \| \leq L\ (\bar{L}_k + 2L_k), n, k = 1, 2, \cdots . \quad \dots (3.9)$$

Thus by virtue of (H_3), it follows from (3.4), (3.7), (3.8) and (3.9) that $\{w_n(t)\}$ is equicontinuous on each $J_k (k = 1, 2, \dots)$. Hence in view of (3.6), an application of Arzela Ascoli's theorem implies that $\left\{ w_n(t) \right\}_{t \in J_k}$ is relatively compact in $C[J_k, E]$, and hence by employing the diagonal method, we show that there exists a subsequence of $\{w_n(t)\}$, converging on each J_k to $w_*(t)$. Since K is normal and $\{w_n(t)\}$ is increasing, the whole sequence $\{w_n(t)\}$ converges uniformly on each J_k to $w_*(t)$. Hence, $w_* \in PC[J, E]$, and (3.7) implies $\| w_* \|_B \leq L$. Therefore, $w_* \in BPC[J, E]$. In view of (3.6), the continuity of f and I_k (k = 1, 2, ...) yields that

$$f(s, w_{n-1}, Tw_{n-1}, Sw_{n-1}) + M(s)w_{n-1}(s) - N(s)(Tw_n - Tw_{n-1}) - N_1(s)(Sw_n - Sw_{n-1})$$

$$\to f(s, w_*, Tw_*, Sw_*) + M(s)w_*(s), \text{ as } n \to +\infty, \text{ for } s \in J\ , \quad \dots (3.10)$$

$$I_k(w_{n-1}(t_k)) - L_k(w_n\ (t_k) - w_{n-1}\ (t_k)) \to I_k(w_*(t_k)), \text{ as } n \to +\infty, \text{ for } k = 1, 2, \dots \quad (3.11)$$

Thus again by $(H_1), (H_2)$, we show that $R(s) \in L^1(J)$ and $\sum_{k=1}^{\infty} L\,(\bar{L}_k + 2L_k)$ is convergent. Hence, by passing to limits in $w_n(t) = Aw_{n-1}(t)$ as $n \to +\infty$, observing (3.8)-(3.11), the dominated convergence theorem implies that $w_*(t) = Aw_*(t)$ for $t \in J$. Therefore, we conclude by Lemma 3 that $w_* \in TPC[J, E] \cap C^1\ [J', E]$ is a solution of IFBVP (1.1).

Analogously, we also obtain that $\{v_n(t)\}$ converges uniformly on each J_k to $v^*(t)$, and $v^* \in TPC[J, E] \cap C^1[J', E]$ is also a solution of IFBVP (1.1). Evidently, $w_*, v^* \in [w_0, v_0]$.

Finally, we can easily show by standard arguments that w_*, v^* are the minimal and maximal solutions of IFBVP (1.1), respectively. We complete the proof.

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