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ON THE CARDINALITY OF THE UNIQUE RANGE SETS FOR MEROMORPHIC AND ENTIRE FUNCTIONS

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Let S denote a finite set in extended complex plane $\bar{\mathbb{C}}$, f a non-constant meromorphic function in the plane, and $\bar{E}(f, S) = \bigcup_{a \in S} \{z | f(z) = a, \text{ ignoring multiplicity}\}$. We define $E_1(f, S) = \{\text{the simple zeroes of } f(z) - s_i, s_i \in S\}$. Recently, Reinders[9] proved that one can exhibit some set S_1 with 16 elements such that the condition $\bar{E}(f, S_1) = \bar{E}(g, S_1)$ implies $f = g$ for any pair of non-constant meromorphic functions f and g, and some set S_2 with 10 elements such that the condition $\bar{E}(f, S_2) = \bar{E}(g, S_2)$ implies $f \equiv g$ for any pair of non-constant entire functions f and g. In this paper, we prove that there is some set S with 15 elements such that the condition $E_1(f, S) = E_1(g, S)$ implies $f = g$ for any pair of non-constant meromorphic functions f and g, and some set T with 9 elements such that the condition $E_1(f, T) = E_1(g, T)$ implies $f \equiv g$ for any pair of non-constant entire functions f and g.

Key Words : Meromorphic Function; Zero; Poles; Share Value; Unique Range Set.

1. INTRODUCTION

By a meromorphic function we shall always mean a function that is analytic in the complex plane $\mathbb{C}$ except at possibly a countable number of poles. Let S be a finite subset of $\bar{\mathbb{C}}$. For a meromorphic function f, let us recall the notations (see, e.g., [4] or [5]):

$$E(f, S) = \bigcup_{a \in S} \{z | f(z) = a, \text{ counting multiplicity}\},$$

$$\bar{E}(f, S) = \bigcup_{a \in S} \{z | f(z) = a, \text{ ignoring multiplicity}\}$$

and define

$$E_1(f, S) = \{\text{all the simple zeroes of } f(z) - s_i, s_i \in S\}.$$

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About seventy years ago, Nevanlinna[7&8] proved two general results: (1). If two non-constant meromorphic functions f and g satisfy $\overline{E}(f, a_i) = \overline{E}(g, a_i)$ $(i = 1, ..., 5)$, where $a_i (i = 1, ..., 5)$ are five distinct points in $\mathbb{C}$, then $f \equiv g$. (2). If f and g are non-constant meromorphic functions satisfying $\overline{E}(f, a_i) = \overline{E}(g, a_i)$ for distinct four values a_1, a_2, a_3, and a_4, then f is a Möbius transformation of g. We say that two non-constant meromorphic functions f and g share a set of values S CM (IM) if and only if $E(f, S) = E(g, S)$ $(\overline{E}(f, S) = \overline{E}(g, (S))$. In [2] the set S such that for any two nonconstant entire functions f and g the condition $E(f, S) = E(g, S)$ implies $f = g$ is called a unique range set for entire functions (URSE in brief). Similarly, one can define a set S as a unique range set for meromorphic functions (URSM), if for any two non-constant meromorphic functions f and g the condition $E(f, S) = E(g, S)$ implies $f = g$. A set S is called a unique range set for entire functions in IM(URSE in IM), if for any two non-constant entire functions f and g the condition $\overline{E}(f, S) = \overline{E}(g, S)$ implies $f = g$. One can similarly define the unique range set for meromorphic functions in IM (URSM in IM). We recall some notations

$$\lambda_M = inf\ \{(S) \mid S \text{ is an URSM}\},\ \lambda_E = \inf\ \{\#(S) \mid S \text{ is an URSM}\}$$

where #(S) is the cardinality of the set S. And we introduce the new notations:

$$\sigma_M = inf\ \{\#(S) \mid S \text{ is an URSM in IM}\},\ \sigma_E = inf\ \{\#(S) \mid S \text{ is an URSE in IM}\}$$

Recently, several papers have been published concerning the smallest $\lambda_M, \lambda_E, \sigma_M$, and σ_E, see, for instances, Yi[11,12], Li-Yang[4,5], and Mues-Reinders[6]. Most recently, Frank-Reinders[1] showed that $\lambda_M \le 11$, and Reinders[9] showed that $\sigma_M \le 16$, $\sigma_E \le 10$. As an attempt to reduce these cardinals furthere, in this paper, we prove, by utilizing Frank-Reinder's argument[1], that there are some set S with 15 elements such that the condition $E_1(f, S) = E_1(g, S)$ implies $f = g$, for any pair of non-constant meromorphic functions f and g, and some set T with 9 elements such that the condition $E_1(f, T) = E_1(g, T)$ implies $f \equiv g$, for any pair of non-constant entire functions f and g. it assumed that the readers are familiar with the standard notations and basic results of the Nevanlinna theory (see, e.g., [3], [10]).

2. Lemmas and Main Results

First, we prove the following Lemma on two functions F and G satisfying $E_1(F, 0) = E_1(G, 0)$ which is the key to the proof of our main result, Theorem 1.

Lemma 1 — Let F and G be non-constant meromorphic functions satisfying $E_1(F, 0) = E_1(G, 0)$ and $c_1, c_2, \ldots, c_q$ be $q(\ge 2)$ distinct non-zero complex numbers. If

$$\limsup_{\substack{r \to \infty \\ r \in I}} \frac{3\overline{N}(r, F) + \sum_{j=1}^{g} \left\{ \overline{N}\left(r, \frac{1}{F - c_j}\right) + \overline{N}\left(2\ r, \frac{1}{F - c_j}\right) \right\} + \overline{N}\left(r, \frac{1}{F}'\right) + 2\overline{N}\left(2\ r, \frac{1}{F}\right)}{R(r, F)} < q \quad \ldots (1)$$

and

$$\limsup_{\substack{r \to \infty \\ r \in I}} \frac{3\overline{N}(r, G) + \sum_{j=1}^{q} \left\{ \overline{N}\left(r, \frac{1}{G-c_j}\right) + \overline{N}_{(2}\left(r, \frac{1}{G-c_j}\right) \right\} + \overline{N}\left(r, \frac{1}{G'}\right) + 2\overline{N}_{(2}\left(r, \frac{1}{G}\right)}{R(r, G)} < q \quad , \quad \dots (2)$$

where $\overline{N}_{(2}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{F}\right) - \overline{N}_{1)}\left(r, \frac{1}{F}\right), \overline{N}_{1)}\left(r, \frac{1}{F}\right)$ is the counting function which only counts the simple zeros of $F(z)$ in $\{z : |z| \leq r\}$ and I is some set of r of infinite linear measure, then

$$F = \frac{aG+b}{cG+d},$$

where $a, b, c, d \in \mathbb{C}$ are constants with $ad - bc \neq 0$.

PROOF : Let

$$H = \frac{F''}{F'} - 2\frac{F'}{F} - \frac{G''}{G'} + 2\frac{G'}{G}. \quad \dots (3)$$

We want to show that $H \equiv 0$. Assuming that $H \not\equiv 0$, the lemma of the logarithmic derivative gives

$$m(r, H) = S(r), \quad \dots (4)$$

where $S(r) = o\ \{T(r)\}$, and $T(r) = max\ \{T(r, F), T(r, G)\}$. Note $E_1(F, 0) = E_1(G, 0)$, and by an elementary calculation, we can conclude that if z_0 is a common simple zero of F and G, then $H(z_0) = 0$. Thus

$$N_{1)}(r) \leq N\left(r, \frac{1}{H}\right) \leq T(r, H) + O(1) \leq N(r, H) + S(r), \quad \dots (5)$$

where $N_{1)}(r) = N_{1)}\left(r, \frac{1}{F}\right) = N_{1)}\left(r, \frac{1}{G}\right)$. The poles of H can only occur at zeros of F' and G' or poles of F and G. Moreover, H can have only simple zeros. We have from (5)

$$N_{1)}(r) \leq \overline{N}(r, F) + \overline{N}(r, G) \mid \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) + \sum_{j=1}^{q} \overline{N}$$

$$_{(2}\left(r, \frac{1}{F-c_j}\right) + \sum_{j=1}^{q} \overline{N}_{(2}\left(r, \frac{1}{G-c_j}\right) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right), \quad \dots (6)$$

where $\overline{N}_0\left(r, \frac{1}{F'}\right)$ is the reduced counting function for the zeros of F' where F does not take one of the values $0, c_1, \dots, c_q$.

On the other hand, we have

$$\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)=2N_1(r)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right\} \quad \dots (7)$$

We obtain from (6) and (7)

$$\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\leq 2\overline{N}(r,F)=2\overline{N}(r,G)+2\overline{N}_0\left(r,\frac{1}{F'}\right)+2\overline{N}_0\left(r,\frac{1}{G'}\right)+$$

$$2\sum_{j=1}^{q}\overline{N}_{(2}\left(r,\frac{1}{F-c_j}\right)+2\sum_{j=1}^{q}\overline{N}_{(2}\left(r,\frac{1}{G-c_j}\right)+3\overline{N}_{(2}\left(r,\frac{1}{F}\right)+3\,\overline{N}_{(2}\left(r,\frac{1}{g}\right\} \dots (8)$$

By the second fundamental theorem, we have

$$qT(r,F)\leq\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{F-c_j}\right)-N_0\left(r,\frac{1}{F'}\right)+S(r),\ r\notin E, \quad \dots (9)$$

$$qT(r,G)\leq\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{G-c_j}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r),\ r\notin E, \dots (10)$$

where E is a set of r of finite linear measure, and it needs not be the same at each occurrence. Combining (8), (9), (10), we obtain for , $r\notin E$,

$$q\{T(r,F)+T(r,G)\}\ \leq 3\overline{N}(r,F)+3\overline{N}(r,G)+\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{F,c_j}\right)+\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{G-c_j}\right)$$

$$+\ 2\sum_{j=1}^{q}\overline{N}_{(2}\left(r,\frac{1}{F-c_j}\right)+2\sum_{j=1}^{q}\overline{N}_{(2}\left(r,\frac{1}{G-c_j}\right)+2\overline{N}_{(2}\left(r,\frac{1}{F}\right)+2\overline{N}_{(2}\left(r,\frac{1}{G}\right)$$

$$+\ \overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+S(r)$$

$$=\ 3\overline{N}(r,F)=3\overline{N}(r,G)+\sum_{j=1}^{q}\left\{\overline{N}\left(r,\frac{1}{F-c_j}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-c_j}\right)\right\}+$$

$$\sum_{j=1}^{q}\left\{\overline{N}\left(r,\frac{1}{G-c_j}\right)+\overline{N}_{(2}\left(r,\frac{1}{G-c_j}\right)\right\}+2\overline{N}_{(2}\left(r,\frac{1}{F}\right)+2\overline{N}_{(2}\left(r,\frac{1}{G}\right)$$

$$+\left\{\sum_{j=1}^{q}\overline{N}_{(2}\left(r,\frac{1}{F-c_j}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)\right\}+$$

$$\left\{\sum_{j=1}^{q} \overline{N}_{(2}\left(r, \frac{1}{G-c_j}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right)\right\} + S(r). \qquad \text{... (11)}$$

But on the other hand, we have

$$\sum_{j=1}^{q} \overline{N}_{(2}\left(r, \frac{1}{F-c_j}\right) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) = \overline{N}\left(r, \frac{1}{F'}\right) \qquad \text{... (12)}$$

and

$$\sum_{j=1}^{q} \overline{N}_{(2}\left(r, \frac{1}{G-c_j}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) = \overline{N}\left(r, \frac{1}{G'}\right) \qquad \text{... (13)}$$

From (11), (12), (13) we have for $r \notin E$

$$\begin{aligned} q\{T(r, F) + T(r, G)\} \leq & \sum_{j=1}^{q} \left\{\overline{N}\left(r, \frac{1}{F-c_j}\right) + \overline{N}_{(2}\left(r, \frac{1}{F-c_j}\right)\right\} \\ & + \sum_{j=1}^{q} \left\{\overline{N}\left(r, \frac{1}{G-c_j}\right) + \overline{N}_{(2}\left(r, \frac{1}{G-c_j}\right)\right\} \\ & + \overline{N}\left(r, \frac{1}{F'}\right) + 3\overline{N}(r, F) + 3\overline{N}(r, G) \\ & + \overline{N}\left(r, \frac{1}{G'}\right) + 2\overline{N}_{(2}\left((r, \frac{1}{F}\right) + 2\overline{N}_{(2}\left(r, \frac{1}{G}\right) + S(r). \end{aligned} \qquad \text{... (14)}$$

From (1), (2), and (14), we derive

$$T(r, F) + T(r, G) \leq o\ \{T(r, F) + T(r, G)\},\ r \in I, r \notin E, \qquad \text{... (15)}$$

which is impossible. Therefore, $H(z) \equiv 0$. i.e.,

$$\frac{F''}{F'} - 2\frac{F'}{F} - \frac{G''}{G'} + 2\frac{G'}{G} \equiv 0.$$

From this we can obtain, by integration,

$$F \equiv \frac{aG = b}{cG + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Lemma 2 — (Yi[11]) — Let F be a non-constant meromorphic function. Then

$$N\left(r, \frac{1}{F^{(n)}}\right) \leq N\left(r, \frac{1}{F}\right) + n\,\overline{N}(r, F) = S(r, F). \qquad \text{... (16)}$$

The following result can be derived from the proof of Frank-Reinder's main result in [1] implicitly.

Lemma 3 — Let $n \geq 6$ and

$$p(w) = \frac{(n-1)(n-2)}{2} w^n - n(n-2)w^{n-1} + \frac{n(n-1)}{2} w^{n-2}.$$

Then $P(w)$ is a unique polynomial for meromorphic functions, i.e. for any two non-constant meromorphic functions f and g, $P(f) \equiv P(g)$ implies $f \equiv g$.

Combining the above lemmas and adapting Frank-Reinders' argument used in [1], we now can prove our main result as follows.

Theorem 1 — *Let n be an integer ≥ 15 and c be a complex number $\neq 0, 1$,*

$$P(w) = \frac{(n-1)(n-2)}{2} w^n - n(n-2) w^{n-1} + \frac{n(n-1)}{2} w^{n-2} - c$$

and

$$S = \{w \in C \mid P(w) = 0)\}.$$

If f and g are two non-constant meromorphic functions satisfying $E_1(f, S) = E_1(g, S)$, then $f \equiv g$.

PROOF : Since

$$P'(w) = \frac{n(n-1)(n-2)}{2} (w-1)^2 w^{n-3} \qquad \text{... (17)}$$

and $P(1) = 1 - c = c_1 \neq 0, P(0) = -c = c_2 \neq 0,$ we have

$$P(w) - c_1 = (w-1)^3 Q_1(w), Q_1(1) \neq 0 \qquad \text{... (18)}$$

and
$$P(w) - c_2 = w^{n-2} Q_2(w), Q_2(0) \neq 0, \qquad \text{... (19)}$$

where Q_1, Q_2 are polynomials of degree n-3 and 2, respectively. Note all the $Q_i (i = 1, 2)$ and P have only simple zeros.

Let F and G be defined as

$$F = P(f), \ G = P(g).$$

Since $E_1(f, S) = E_1(g, S)$, we have $E_1(F, 0) = E_1(G, 0)$. From (18), and (19), we have

$$\overline{N}\left(r, \frac{1}{F-c_1}\right) + \overline{N}_{(2}\left(r, \frac{1}{F-c_1}\right) \leq 2\,\overline{n}\left(r, \frac{1}{f-1}\right) + \sum_{i=1}^{n-3} N\left(r, \frac{1}{f-a_i}\right)$$

$$\leq (n-1)\, T(r, f) + O(1), \qquad \text{... (20)}$$

$$\overline{N}\left(r,\frac{1}{F-c_2}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-c_2}\right)\le 2\overline{N}\left(r,\frac{1}{f}\right)+\sum_{j=1}^{2}N\left(r,\frac{1}{f-b_j}\right)\le 4T(r,f)+O(1), \quad \text{... (21)}$$

where a_i (i = 1, ..., n − 3) and b_j (j = 1, 2) are the zeros of $Q_1(w)$ and $Q_2(w)$, respectively. From (17), we have

$$\overline{N}\left(r\frac{1}{F'}\right)\le\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{f'}\right). \quad \text{... (22)}$$

Again from (17), (18) and (19), we have

$$\overline{N}_{(2}\left(r,\frac{1}{F}\right)=\sum_{i=1}^{n}\overline{N}\left(r,\frac{1}{f-d_i}\right)\le\overline{N}\left(r,\frac{1}{f'}\right), \quad \text{... (23)}$$

where d_i (i = 1, ..., n) are the distinct zeros of $P(w)$. By (22), (23), and lemma 2 we have

$$\overline{N}\left(r,\frac{1}{F'}\right)+2\overline{N}_{(2}\left(r,\frac{1}{F}\right)\le\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+3N\left(r,\frac{1}{f}\right)+3\overline{N}(r,f)$$

$$\le 5T(r,f)+3\overline{N}(r,f) \quad \text{... (24)}$$

By combining (20), (21) and (24), and noting that $T(r,F)=nT(r,f)+O(1)$, we have

$$\limsup_{\substack{r\to\infty\\ r\notin E}}\frac{3\overline{N}(r,F)+\sum_{j=1}^{2}\left\{\overline{N}\left(r,\frac{1}{F-c_j}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-c_j}\right)+\overline{N}\left(\frac{1}{F'}\right)+2\overline{N}_{(2}\left(r,\frac{1}{F}\right)\right.}{T(r,F)} \quad \text{... (25)}$$

$$\le\limsup_{\substack{r\to\infty\\ r\notin E}}\frac{6\overline{N}(r,f)+(n+8)\,T(r,f)}{nT(r,f)}<2.$$

$$\limsup_{\substack{r\to\infty\\ r\notin E}}\frac{3\overline{N}(r,G)+\sum_{j=1}^{2}\left\{\overline{N}\left(r,\frac{1}{G-c_j}\right)+\overline{N}_{(2}\left(r,\frac{1}{G-c_j}\right)+\overline{N}\left(\frac{1}{G'}\right)+2\overline{N}_{(2}\left(r,\frac{1}{G}\right)\right.}{T(r,G)} \quad \text{... (26)}$$

$$\le\limsup_{\substack{r\to\infty\\ r\notin E}}\frac{6\overline{N}(r,g)+(n+8)\,T(r,g)}{nT(r,g)}<2.$$

Thus by Lemma 1, we have

$$F=\frac{aG+b}{cG+d},$$

where $a,b,c,d\in\mathbb{C}$ are constants with $ad-bc\neq 0$. Note that $E_1(f,S)$ is non-empty, and since $E_1(f,S)=E_1(g,S)$, we have b = 0, $a\neq 0$. Hence

$$F \equiv \frac{aG}{cG+d} \equiv \frac{G}{AG+B},$$

where $A = \frac{c}{a}, B = \frac{d}{a} \neq 0$. Now we discussion two cases: $A \neq 0$ and $A = 0$, separately.

Case 1 — $A \neq 0$. It follows that every zero of $P(g) + \frac{B}{A}$ has a multiplity of at least n. Here are three subcases to be investigated. (1) $\frac{B}{A} = -c_1$, (2) $\frac{B}{A} = -2$, and (3) $\frac{B}{A} \neq -c_1, -c_2$.

If $\frac{B}{A} = -c_1$, we have from (18)

$$P(g) + \frac{B}{A} = (g-1)^3 (g-a_1) \ldots (g-a_{n-3}),$$

where $a_i \neq 0, 1$, are distinct values. This means that $g(z)$ has n-2 completely ramified values. According to second fundamental theorem we can have only 4 such kind of values, i.e. $n-2 \leq 4$. It follows that $n \leq 6$ which contradicts with the assumption that $n \geq 15$. If $\frac{B}{A} = -c_2$, we have from (19)

$$P(g) + \frac{B}{A} = g^{n-2}(g-b_1)(g-b_2),$$

where $b_1 \neq b_2, b_i \neq 0$, 1 (i = 1, 2). Hence, every zero of g has a multiplicity at least 2 and every zero of $g - b_i$ (i = 1, 2) has a multiplicity at least n. So, again, by second fundamental theorem we have

$$T(r,g) < \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-b_1}\right) + \overline{N}\left(r, \frac{1}{g-b_2}\right) + S(r,g)$$

$$\leq \frac{1}{2}N\left(r, \frac{1}{g}\right) + \frac{1}{n}N\left(r, \frac{1}{g-b_1}\right) + \frac{1}{n}N\left(r, \frac{1}{g-b_2}\right) + S(r,g) \leq \left(\frac{1}{2} + \frac{2}{n}\right)T(r,g) + S(r,g).$$

When $n \geq 5$, this is impossible.

We can similarly discuss the case $\frac{B}{A} \neq -c_1 - c_2$ and derive the same contradication.

Case 2 — $A = 0$. If $B \neq 1$, then from (18) we have

$$P(f) - \frac{c_2}{B} = \frac{1}{B}(P(g) - c_2) = \frac{1}{B}\{g^{n-2}(g-b_1)(g-b_2)\}. \qquad \ldots (27)$$

Noting $\frac{c_2}{B} \neq c_2, P(f) - \frac{c_2}{B}$ has at least n-2 distinct zeros $\alpha_1, \ldots, \alpha_{n-2}$, and applying second fundamental theorem again we have

$$(n-4)\,T(r,f) \le \sum_{i=1}^{n-2} \overline{N}\left(r, \frac{1}{f-\alpha_i}\right) + S(r,f)$$

$$\le \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-b_1}\right) + \overline{N}\left(r, \frac{1}{g-b_2}\right) + S(r,f)$$

$$\le 3T(r,g) + S(r,f) = 3T(r,g) + S(r,f).$$

It follows from this and (27) that $n \le 7$ which is a contradiction.

Therefore, we have $A = 0$ and $B = 1$, i.e. $P(f) = P(g)$. By Lemma 3 we have $f \equiv g$. This completes the proof.

Theorem 2 — *Let $n \ge 9$ be an integer and $c \ne 0$, 1 a complex number,*

$$P(w) = \frac{(n-1)(n-2)}{2} w^n - n(n-2)w^{n-1} + \frac{n(n-1)}{2} w^{n-2} - c$$

and

$$S = \{w \in C \mid P(w) = 0\}.$$

Assume f and g are two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{5}{6}$ and $\Theta(\infty, g) > \frac{5}{6}$. If $E_1(f, S) = E_1(g, S)$, then $f \equiv g$.

PROOF : Since $\Theta(\infty, f) > \frac{5}{6}$ and $\Theta(\infty, g) > \frac{5}{6}$, we have

$$\limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} < \frac{1}{6}, \quad \limsup_{r\to\infty} \frac{\overline{N}(r,g)}{T(r,g)} < \frac{1}{6}. \qquad \dots (28)$$

From (25) and (26) we have, respectively,

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{3\overline{N}(r,F) + \sum_{j=1}^{2}\left\{\overline{N}\left(r, \frac{1}{F-c_j}\right) + \overline{N}_{(2}\left(r, \frac{1}{F-c_j}\right)\right\} + \overline{N}\left(r, \frac{1}{F'}\right) + 2\overline{N}_{(2}\left(r, \frac{1}{F}\right)}{T(r,F)} \qquad \dots (29)$$

$$\le \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{6\overline{N}(r,f) + (n+8)\,T(r,f)}{nT(r,f)} < 2$$

and

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{3\overline{N}(r,G) + \sum_{j=1}^{2}\left\{\overline{N}\left(r, \frac{1}{G-c_j}\right) + \overline{N}_{(2}\left(r, \frac{1}{G-c_j}\right)\right\} + \overline{N}\left(r, \frac{1}{G'}\right) + 2\overline{N}_{(2}\left(r, \frac{1}{G}\right)}{T(r,G)}$$

$$\leq \limsup_{\substack{r \to \infty \\ r \notin E}} \frac{6\overline{N}(r, g) + (n+8)\, T(r, g)}{nT(r, g)} < 2.$$

Thus Lemma 1 is applicable and hence $F = \frac{aG+b}{cG=d}$. By using arguments similar to that in the proof of Theorem 1, we obtain $f \equiv g$

From this we have immediately the following result.

Corollary — Let $n \geq 9$, $P(w)$ and S be the same as defined in Theorem 2. Assume f and g are two non-constant entire functions. If $E_1(f, S) = E_1(g, S)$, then $f \equiv g$.

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