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THE NONSPLIT DOMINATION NUMBER OF A GRAPH

V. R. KULLI AND B. JANAKIRAM

Department of Mathematics, Gulbarga University, Gulbarga 585 106

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A dominating set D of a graph $G = (V, E)$ is a nonsplit dominating set if the induced subgraph $\langle V-D \rangle$ is connected. The nonsplit domination number $\gamma_{ns}(G)$ of G is the minimum cardinality of a nonsplit dominating set. In this paper, many bounds on $\gamma_{ns}(G)$ are obtained and its exact values for some standard graphs are found. Also, its relationship with other parameters is investigated.

Key Words : Graph; Domination number; Nonsplit Domination Number

1. INTRODUCTION

The graphs considered here are finite, undirected nontrivial and connected without loops or multiple edges.

Let $G = (V, E)$ be a graph. A set $D \subset V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set.

A dominating set D of G is a connected dominating set if the induced subgraph $\langle D \rangle$ is connected. The connected domination number $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set.

Recently, Kulli and Janakiram introduced the concept of split domination in [5].

A dominating set D of a graph $G = (V, E)$ is a split dominating set if the induced subgraph $\langle V-D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set.

The reader is referred to [1], [2] and [3] for survey or results on domination.

Any undefined term in this paper may be found in Harary[4]. Unless stated, the graph has p vertices and q edges.

The purpose of this paper is to introduce the concept of Nonsplit Domination.

A dominating set D of a graph $G = (V, E)$ is a nonsplit dominating set if the induced subgraph $\langle V-D \rangle$is connected. The nonsplit domination number $\gamma_{ns}(G)$ of G is the minimum cardinality of a nonsplit dominating set.

We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$. Similarly, a γ_c-set, a γ_s-set and a γ_{ns}-set are defined.

2. RESULTS

We start with some elementary results. Since their proofs are trivial, we omit the same.

***Theorem* 1** — *For any graph G,*

$$\gamma(G) \leq \gamma_{ns}(G). \quad \text{... (1)}$$

***Theorem* 2** — *For any graph G,*

$$\gamma(G) = \min \{\gamma_s(G), \gamma_{ns}(G)\}. \quad \text{... (2)}$$

In [3], Cockayne and Hedetniemi gave necessary and sufficient conditions for a minimal dominating set.

***Theorem* A[3]** — *A dominating set D of a graph G is minimal if and only if for each vertex $v \in D$ one of the following conditions is satisfied:*

(i) there exists a vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$; and

(ii) v is an isolated vertex in $\langle D \rangle$.

***Theorem* 3** — *A nonsplit dominating set D of G is minimal if and only if for each vertex $v \in D$ one of the following conditions is satisfied :*

(i) there exists a vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$;

(ii) v is an isolated vertex in $\langle D \rangle$; and

(iii) $N(v) \cap (V - D) = \phi$.

PROOF : Suppose D is minimal. On the contrary, if there exists a vertex $v \in D$ such that v does not satisfy any of the given conditions, then by Theorem A, $D' = D - \{v\}$ is a dominating set of G and by (*iii*), $\langle V - D' \rangle$ is connected. This implies that D' is a nonsplit dominating set of G, a contradiction. This proves the necessity.

Sufficiency is straightforward.

Next we obtain a relationship between $\gamma_{ns}(G)$ and $\gamma_{ns}(H)$ where H is any spanning subgraph of G. We omit the proof.

Theorem 4 — *For any spanning subgraph H of G,*

$$\gamma_{ns}(G) \leq \gamma_{ns}(H). \quad \text{... (3)}$$

In the following two results, we obtain lower and upper bounds on $\gamma_{ns}(G)$ respectively.

***Theorem* 5** — *For any graph G,*

$$\gamma_{ns} G \geq (2p - q - 1)/2 \quad \text{... (4)}$$

PROOF : Let D be a γ_{ns}-set of G. Since $\langle v-D \rangle$ is connected.

$$q \geq |V-D| + |V-D| - 1.$$

This proves (4).

***Theorem* 6** — *For any graph G,*

$$\gamma_{ns}(G) \leq p - \omega(G) + 1, \quad \ldots (5)$$

where $\omega(G)$ is the clique number of G.

PROOF : Let S be a set of vertices of G such that $\langle S \rangle$ is complete with $|S| = \omega(G)$. Then for any $u \in S$, $(V-S) \cup \{u\}$ is a nonsplit dominating set of G.

Thus (5) holds.

Now we list the exact values of $\gamma_{ns}(G)$ for some standard graphs.

Proposition 7 —

(*i*) *For any complete graph K_p with $p \geq 2$ vertices,*

$$\gamma_{ns}(K_p) = 1. \quad \ldots (6)$$

(*ii*) *For any complete bipartite graph $K_{m,n}$ with $2 \leq m \leq n$,*

$$\gamma_{ns}(K_{m,n}) = 2. \quad \ldots (7)$$

(*iii*) *For any cycle C_p,*

$$\gamma_{ns}(C_p) = p - 2. \quad \ldots (8)$$

(*iv*) *For any wheel W_p,*

$$\gamma_{ns}(W_p) = 1. \quad \ldots (9)$$

(*v*) *For any path P_p with $p \geq 3$ vertices,*

$$\gamma_{ns}(P_p) = p - 2. \quad \ldots (10)$$

Our next result sharpens the inequality (5) for trees.

***Theorem* 8** — *If T is a tree which is not a star, then,*

$$\gamma_{ns}(T) \le p-2. \quad \text{... (11)}$$

PROOF : Since T is not a star, there exist two adjacent cut vertices u and v with deg u, deg $v \ge 2$. This implies that $V-\{u, v\}$ is a nonsplit dominating set of T.

Thus (11) holds.

***Theorem* 9** — *If* $\kappa(G) > \beta_0(G)$, *then*

$$\gamma_{ns}(G) = \gamma(G), \quad \text{... (12)}$$

where $\kappa(G)$ *is the connectivity of G and* $\beta_0(G)$ *is the independence number of G.*

PROOF : Let D be a γ-set of G. Since $\kappa(G) > \beta_0(G) \ge \gamma(G)$, it implies that $\langle V-D \rangle$ is connected. This proves that D is a γ_{ns}-set of G. Hence (12) follows.

***Theorem* 10** — *Let D be a* γ_{ns}*-set of a connected graph G. If no two vertices in V-D are adjacent to a common vertex in D, then*

$$\gamma_{ns}(G) + \varepsilon(T) \ge p \quad \text{... (13)}$$

where $\varepsilon(T)$ *is the maximum number of endvertices in any spanning tree T of G.*

PROOF : Let D be a γ_{ns}-set of G, given in the hypothesis. Since for any two vertices $u, v \in V-D$, there exist two vertices $u_1, v_1 \in D$ such that u_1 is adjacent to u but not to v and v_1 is adjacent to v but not to u_1, this implies that there exists a spanning tree T of $\langle V-D \rangle$ in which each vertex of $V-D$ is adjacent to a vertex of D. This proves that $\varepsilon(T) \ge |V-D|$.

Thus (13) holds.

***Theorem* 11** — *If* $\delta(G) + \omega(G) \ge p+1$, *then*

$$\gamma_c(G) + \gamma_{ns}(G) \le p \quad \text{... (14)}$$

where $\delta(G)$ *is the minimum degree of G.*

PROOF : By (5), $\gamma_{ns}(G) \le p - \omega(G) + 1$

$$\le \delta(G).$$

Let D be a γ_{ns}-set of G. Then every vertex in D is adjacent to some vertex in $V-D$. Thus $\langle V-D \rangle$ is a connected dominating set of G, since $\langle V-D \rangle$ is connected. This proves (14).

In the next result we obtain another upper bound on $\gamma_{ns}(G)$.

***Theorem* 12** — *For any graph G,*

$$\gamma_{ns}(G) \le - \text{diam}(G) + h + 1, \quad \text{... (15)}$$

where diam(G) is the diameter of G and h is the minimum number of vertices in a γ_{ns}-set of G which lie in between shortest u-v path and d(u, v) diam (G)

PROOF : Let diam(G) = k. We consider the following cases.

Case 1 — Suppose $u, v \in V-D$. Then $V - D$ has at least k+1 vertices.

Case 2 — Suppose $u \in D$ and $v \in V-D$. If there exists a vertex $u_1 \in V-D$ such that u_1 is connected to u through the vertices of D then, $d(u_1, v) \geq k-(h+1)$ and hence $V - D$ has at least $k - h$ vertices. For otherwise, for every vertex $u_1 \in V-D$ there exists a vertex w adjacent to u_1 such that $d(u, w) = d(u, v) + d(v, u_1) + d*(u_1, w) \geq k+1$, a contradiction.

This implies that $V - D = \{v\}$ and hence $G = K_2$ or $K_{1,2}$.

Case 3 — Suppose u, v D. If there exist two vertices $u_1, v_1 \in V-D$ such that u is connected to u_1 and v is connected to v_1 through the vertices of D, then $d(u_1, v_1) \geq k-(h+2)$ and hence V–D has at least k–h–1 vertices. For otherwise, there exists exactly one vertex $u_1 \in V-D$ which is adjacent to both u and v and $\{u_1\}$ = V–D. This implies that G is a star with at least three vertices.

Thus from the above all the three cases, it follows that V–D has at least k–h–1 vertices and hence (15) follows.

Now we obtain a lower bound on $\gamma_{ns}(T)$.

***Theorem* 13** — *For any tree T,*

$$\gamma_{ns}(T) \geq p-m, \qquad \text{... (16)}$$

where m is the number of vertices adjacent to endvertices.

PROOF : If T is K_2, the result is trivial. If T has at least three vertices and D is a γ_{ns}-set of T, then each vertex of V-D is a cutvertex of T. Let S be the set of all cutvertices which are adjacent to endvertices with $|S| = m$. Let $u \in V-D$. If $u \in S$, then D = V–S and (16) holds. If $u \notin S$, then there exists a cutvertex $v \in D$ adjacent to u. Further, all vertices which are connected to v not through u also belonging to D. This implies that V-D has at most m vertices and (16) holds.

Corollary 13.1 — *For any tree T,*

$$\gamma_c(T) \leq \gamma_{ns}(T). \qquad \text{... (17)}$$

Further if T is a path, then equality holds.

PROOF : If T has no cut vertices, then $T = K_2$ and hence $\gamma_c(T) = \gamma_{ns}(T) = 1$.

Let S be the set of all cut vertices of T with $|S| = p_1$ and $S_1 \subseteq S$ be the set of all cut vertices such that each vertex of S_1 is adjacent to an endvertex with $|S_1| = p_2$.

Thus,

$$V(T) = p \geq p_1 + p_2.$$

Due to Sampathkumar and Walikar[6],

$$\gamma_c(T) = p_1.$$

Hence, (17) follows from (16).

If T is a path with $p \geq 3$ vertices, then by (10) and the fact that $\gamma_c(T) = p_1$, the equality holds.

Next we obtain an upper bound on $\gamma_{ns}(T)$.

***Theorem* 14** — *For any tree T,*

$$\gamma_{ns}(T) \leq p - \max_{v} \{\deg\ v - |(e(v)|\}, \qquad \text{... (18)}$$

where e(v) is the set of all endvertices adjacent to v.

PROOF : Let v be a vertex with $\deg v - |e(v)|$ being maximum. Let $u \in N(v)$. Then it follows that $V - N[v] \cup e(v) \cup \{u\}$ is a nonsplit dominating set of T. Hence, (18) holds.

Corollary 14.1 — *For any tree T,*

$$\gamma_{ns}(T) \leq p - \Delta(T) + p_0, \qquad \text{... (19)}$$

where $\Delta(T)$ is the maximum degree of T and p_0 is the minimum number of endvertices adjacent to a vertex of maximum degree.

Corollary 14.2 — *For any graph G,*

$$\gamma_{ns}(G) \leq p - \max_{v} \{deg v - |e(v)|\}, \qquad \text{... (20)}$$

where e(v) is the set of all vertices which are adjacent to v but not adjacent to any vertex of V-N(v).

PROOF : This follows from the fact that for any $v \in V$, there exists a spanning tree T such that $deg_G v = deg_T v$ and from (3) and (18).

The next result relates to $\gamma_{ns}(\overline{G})$ and $\gamma_s(\overline{G})$ where $\overline{G}$ is the complement of G.

***Theorem* 15** — *If diam (G) = 5, then*

$$\gamma_s(G) \geq \gamma_{ns}(\overline{G}). \qquad \text{... (21)}$$

PROOF : Let D be a γ_s-set of G. Then every vertex in V-D is not adjacent to at least one vertex in D, since diam (G) = 5. Thus D is a dominating set of $\overline{G}$ and further it is a nonsplit dominating set of $\overline{G}$, as $\langle V - D \rangle$ is connected in $\overline{G}$.

This proves (21).

The following result is obvious. Hence, we omit its proof.

***Theorem* 16** — *Let G be a graph such that both G and $\overline{G}$ are connected. Then*

(i) $\gamma_{ns}(G) + \gamma_{ns}(\overline{G}) \leq 2(p-2)$; ... (22)

and (ii) $\gamma_{ns}(G) \cdot \gamma_{ns}(\overline{G}) \leq (p-2)^2$. ... (23)

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